Resurgence and Non-Perturbative Physics

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Lecture 1

- motivation
- divergence of perturbation theory in QM
- basics of Borel summation
- the Bogomolny/Zinn-Justin cancellation mechanism

Motivation

Resurgence:
- 'new' idea in mathematics
- goal: explore implications for physics

- unification of perturbation theory and non-perturbative physics
- applications to QM, QFT, Strings, ...
- consistent non-perturbative definition of asymptotically free QFT
- insight into localization
- analytic continuation of path integrals
- exponentially improved ('exact') semi-classical analysis

Perturbation theory

- perturbation theory generally produces a divergent series
- semiclassical (WKB) expansions are generally divergent
- there is a lot of interesting physics encoded in these facts
- perturbation theory has nontrivial 'hidden' structure
- perturbation theory and non-perturbative physics are intricately entwined
- "resurgence" describes these inter-relations
- general mathematical approach to *instanton calculus*

Perturbation theory generally produces a divergent series

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever ... That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

N. Abel, 1802 – 1829

The series is divergent; therefore we may be able to do something with it

O. Heaviside, 1850 – 1925

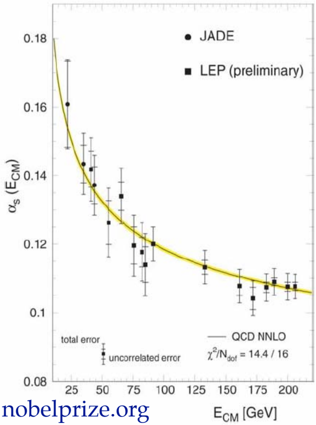
Perturbation theory works

QED perturbation theory:

$$\frac{1}{2}(g-2) = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - (1.7283(35))\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,184\,42$$



nobelprize.org

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$

resurgence = unification of perturbation theory and non-perturbative physics

• cures inconsistencies in perturbative OR non-perturbative analyses

• series expansion $\longrightarrow$ *trans-series* expansion

• trans-series well-defined under analytic continuation of parameter

• philosophical shift:

view semiclassical expansions as potentially exact

• applications: ODEs, PDEs, QM, QFT, String Theory, ...

Resurgent Trans-Series

- trans-series expansion:

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} a_{n,k,l}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\log\left(-\frac{1}{g^2}\right)\right]^l$$

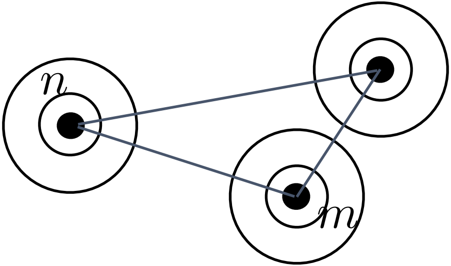
- J. Écalle (1980): set of functions with these trans-monomial elements is closed under:

(Borel transform)+(analytic continuation)+(Laplace transform)

- "any reasonable function" has a trans-series expansion
- differential equations, iterated maps, ...
- trans-series expansion coefficients are highly correlated
- exponentially improved asymptotic expansions

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Divergence of perturbation theory in quantum mechanics

e.g. ground state energy:

$$E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$$

- cubic oscillator: $c_n \sim -\frac{(60)^{n+1/2}}{(2\pi)^{3/2}} \, \Gamma(n+\frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^{n+1} \frac{3^n \sqrt{6}}{\pi^{3/2}} \, \Gamma(n+\frac{1}{2})$
- Zeeman: $c_n \sim (-1)^n \left(\frac{4}{\pi}\right)^{5/2} \frac{1}{\pi^{2n}} \, \left(2n+\frac{1}{2}\right)!$
- Stark: $c_n \sim -\frac{4}{\pi} \left(\frac{3}{2}\right)^{2n+1} (2n)!$
- periodic Sine-Gordon potential: $c_n \sim n!$
- double-well: $c_n \sim 3^n \, n!$

note generic factorial growth of perturbative coefficients

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n \, (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

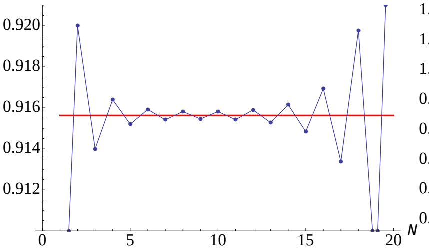
$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

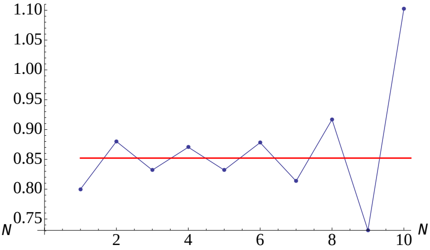
truncate just before least term (x dependent!)

Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, x^n \sim \frac{1}{x} \, e^{\frac{1}{x}} \, E_1 \left(\frac{1}{x} \right)$$



$x = 0.1$

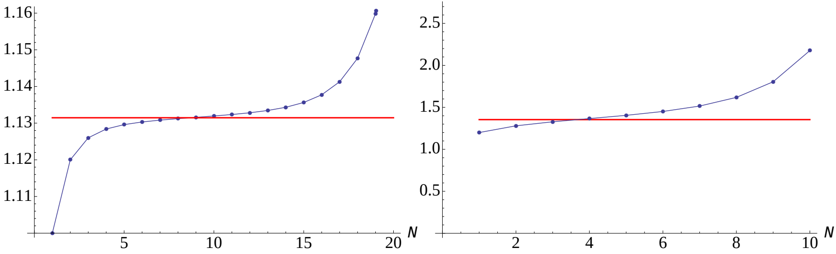


$x = 0.2$

optimal order depends on x: $\quad N \approx \frac{1}{x}$

Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty} n!\, x^n \sim \frac{1}{x}\, e^{-\frac{1}{x}}\, Ei\left(\frac{1}{x}\right)$$



$x = 0.1$ $\quad\quad$ $x = 0.2$

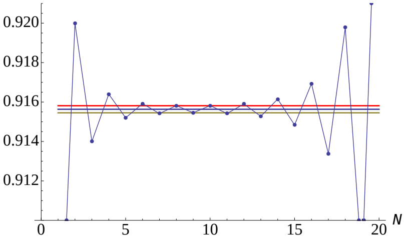
optimal order depends on x: $\quad N \approx \frac{1}{x}$

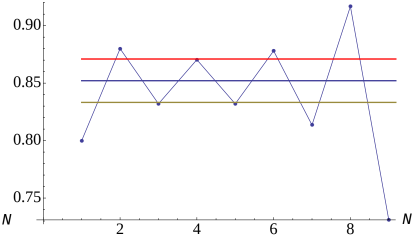
Asymptotic Series vs Convergent Series

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$

optimal truncation: error term is <u>exponentially small</u>

$$\left|R_N(x)\right|_{N\approx 1/x} \approx \left. N!\, x^N \right|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$





$x = 0.1$ $\qquad$ $x = 0.2$

Divergence of perturbation theory in quantum mechanics

typical large order growth:

$$c_n \sim (\pm 1)^n \, \beta^n \, \Gamma(\gamma \, n + \delta)$$

Related to factorial growth of number of Feynman diagrams

$$J \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4} \, dx = \sum_{n=0}^{\infty} J_n \, g^n$$

$$\Rightarrow \quad J_n \sim (-1)^n \, \frac{(n-1)!}{4^n}$$

Borel summation: basic idea

example: exponential integral function
(http://dlmf.nist.gov/6.2)

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, g^n = \frac{1}{g} \, e^{\frac{1}{g}} \, E_1\left(\frac{1}{g}\right)$$

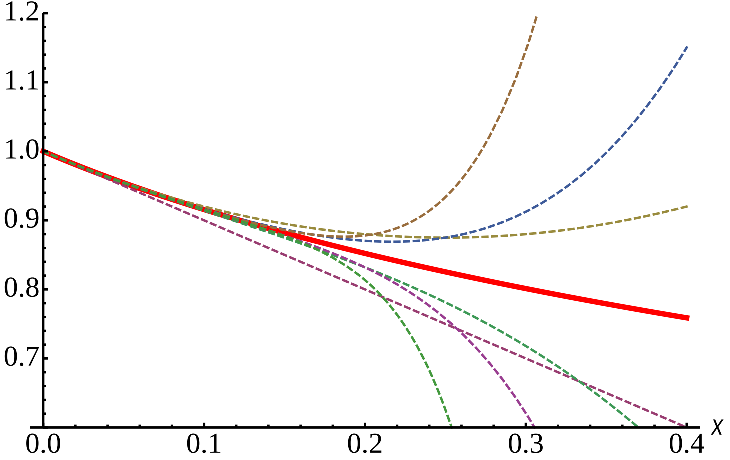
write $n! = \int_0^\infty dt \, e^{-t} \, t^n$

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, g^n = \int_0^\infty dt \, e^{-t} \, \frac{1}{1+g\,t} = \frac{1}{g} \int_0^\infty dt \, e^{-t/g} \, \frac{1}{1+t}$$

integral convergent for all $g > 0$: "Borel sum" of the series

Borel Summation: basic idea

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1+x\,t}$$



Borel summation: basic idea

example: non-alternating series:

$$\sum_{n=0}^{\infty} n!\, g^n = \frac{1}{g}\, e^{-\frac{1}{g}}\, Ei\left(\frac{1}{g}\right)$$

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

$$\sum_{n=0}^{\infty} n!\, g^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1 - g\, t} = \frac{1}{g} \int_0^\infty dt\, e^{-t/g}\, \frac{1}{1-t} \quad ???$$

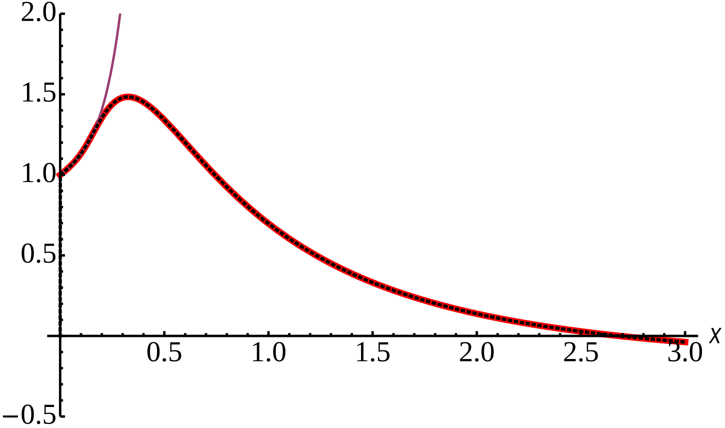
pole on the Borel axis!

$\Rightarrow$ non-perturbative imaginary part

$$\pm \frac{i\,\pi}{g}\, e^{-\frac{1}{g}}$$

Borel Summation: Basic Idea

$$\text{Borel} \quad \Rightarrow \quad \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^\infty dt\, e^{-t}\, \frac{1}{1-x\,t} = \frac{1}{x}\, e^{-\frac{1}{x}}\, Ei\left(\frac{1}{x}\right)$$



Borel summation

Borel transform of series $f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n$:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has finite radius of convergence.

Borel resummation of original asymptotic series:

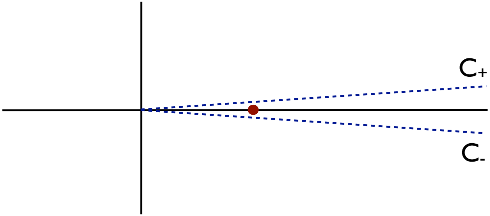
$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

warning: $\mathcal{B}[f](t)$ may have singularities in (Borel) t plane

Borel singularities

avoid singularities on $\mathbb{R}^+$: lateral Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



go above/below the singularity: $\theta = 0^\pm$

$\longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{S}_0 f(g)]$

challenge: use physical input to resolve ambiguity

Divergence of perturbation theory in quantum mechanics

e.g. ground state energy:

$$E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$$

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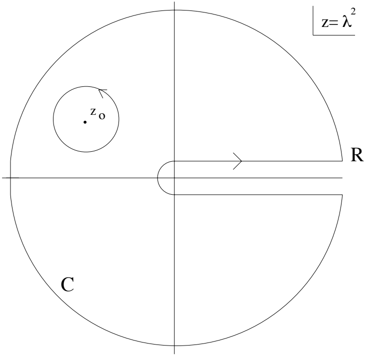


$$E(z_0) = \frac{1}{2\pi i}\oint_C dz\,\frac{E(z)}{z-z_0}$$

$$= \frac{1}{\pi}\int_0^R dz\,\frac{Im\,E(z)}{z-z_0}$$

$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\,\frac{\mathrm{Im}\,E(z)}{z^{n+1}}\right)$$

$$\mathrm{WKB} \Rightarrow \mathrm{Im}\,E(z) \sim \frac{a}{\sqrt{z}}\,e^{-b/z} \quad , \quad z \to 0$$

$$\Rightarrow c_n \sim \frac{a}{\pi}\int_0^\infty dz\,\frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}}$$

Divergence of perturbation theory

an important part of the story ...

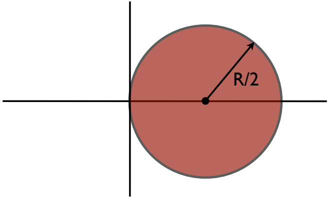
> *The majority of nontrivial theories are seemingly unstable at some phase of the coupling constant, which leads to the asymptotic nature of the perturbative series*

A. Vainshtein (1964)

Borel summation: existence theorem (Nevanlinna & Sokal)

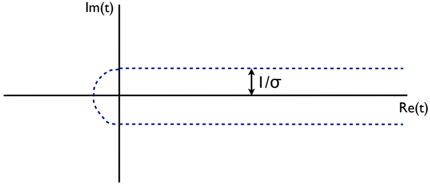
$f(z)$ analytic in circle $C_R = \{z : \left|z - \frac{R}{2}\right| < \frac{R}{2}\}$

$$f(z) = \sum_{n=0}^{N-1} a_n\, z^n + R_N(z) \qquad , \qquad |R_N(z)| \leq A\, \sigma^N\, N!\, |z|^N$$



Borel transform

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\, t^n$$



analytic continuation to
$S_\sigma = \{t : |t - \mathbb{R}^+| < 1/\sigma\}$

$$f(z) = \frac{1}{z} \int_0^\infty e^{-t/z}\, B(t)\, dt$$

Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\, \Gamma(\gamma\, n + \delta)$$

- alternating series: real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{1+t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

- nonalternating series: ambiguous imaginary part

$$\mathrm{Re}\, f(-g) \sim \frac{1}{\gamma}\, \mathcal{P} \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{1-t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

$$\mathrm{Im}\, f(-g) \sim \pm\frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

Borel summation in practice

direct quantitative correspondence between:

rate of growth $\leftrightarrow$ Borel poles $\leftrightarrow$ non-perturbative exponent

non-alternating factorial growth: $c_n \sim \beta^n \, \Gamma(\gamma \, n + \delta)$

positive Borel singularity: $t_c = \left(\frac{1}{\beta \, g}\right)^{1/\gamma}$

non-perturbative exponent: $\pm i \, \frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$

recall: Divergence of perturbation theory in QM

e.g. ground state energy:

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- cubic oscillator: $c_n \sim -\frac{(60)^{n+1/2}}{(2\pi)^{3/2}} \, \Gamma(n+\frac{1}{2})$
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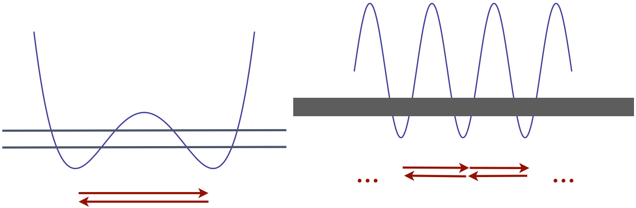
- periodic Sine-Gordon potential: $c_n \sim n!$ stable ???
- double-well: $c_n \sim 3^n \, n!$ stable ???

Bogomolny/Zinn-Justin mechanism in QM



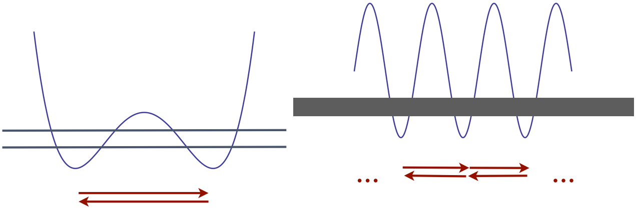
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel summable: ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel!

Bogomolny/Zinn-Justin mechanism in QM

e.g., double-well: $V(x) = x^2(1 - g\,x)^2$

$$E_0 \sim \sum_n c_n\, g^{2n}$$

- perturbation theory:

$$c_n \sim -3^n\, n! \quad \to \quad \mathrm{Im}\, E_0 \sim \mp\, \pi\, e^{-\frac{1}{3g^2}}$$

- non-perturbative instanton gas:

$$\mathrm{Im}\, E_0 \sim \pm\, \pi\, e^{-2\frac{1}{6g^2}}$$

- BZJ cancellation $\Rightarrow$ E_0 is real and unambiguous

"resurgence" $\Rightarrow$ cancellation to all orders

Bogomolny/Zinn-Justin mechanism in QM

- double-well potential: $V(x) = \frac{1}{2}\, x^2\, (1 - g\, x)^2$
- instanton solution: $g\, x_0(t) = 1/(1 + e^{-t})$
- classical Eucidean action: $S_0 = \frac{1}{6g^2}$

approximate $\mathcal{I}\bar{\mathcal{I}}$ soln. : $x_{cl}(t) = \begin{cases} x_0(R+t) & , \quad t > 0 \\ x_0(R-t) & , \quad t < 0 \end{cases}$

effective interaction potential: $U_{\rm int}(t_1, t_2) = -\frac{2}{g^2}\, e^{-|t_1 - t_2|}$

$$
\begin{aligned}
Z_{\rm int} \quad &= \quad a^2 \int dt_1 \int dt_2\, e^{-U_{int}(t_1,t_2)} \qquad \left(a \equiv \frac{1}{g\sqrt{\pi}}\, e^{-\frac{1}{6g^2}} \right) \\
&\overset{T\to\infty}{\sim} \quad \frac{1}{2} T^2\, a^2 + T\, a^2 \int_0^\infty dt \left(\exp\left[\frac{2}{g^2} e^{-t} \right] - 1 \right) + \ldots
\end{aligned}
$$

- as $g^2 \to 0$, dominated by $t \to 0$???

Bogomolny/Zinn-Justin mechanism in QM

$$Z_{\text{int}} \overset{T\to\infty}{\sim} \frac{1}{2}T^2\, a^2 + T\, a^2 \int_0^\infty dt \left(\exp\left[\frac{2}{g^2} e^{-t}\right] - 1 \right) + \ldots$$

BZJ idea: analytically continue $g^2 \to -g^2$

$\Rightarrow$ dominated by finite t $\Rightarrow$ stable instanton gas

$$\int_0^\infty dt \left(\exp\left[-\frac{2}{g^2} e^{-t}\right] - 1 \right) \sim -\gamma_E + \ln\left(\frac{g^2}{2}\right) + Ei\left(-\frac{2}{g^2}\right)$$

- ambiguous imaginary part (from log) when $-g^2 \to g^2$
- recall $Z \sim e^{-E_0\, T}$ $\Rightarrow$ imaginary E_0 from instanton gas

BZJ cancellation: cancels against ambiguous imaginary part from analytic continuation of Borel summation of perturbation theory

Bogomolny/Zinn-Justin mechanism in SUSY QM

Balitsky/Yung: SUSY double-well

$$V_{\text{bosonic}} = W^2 - W' = \frac{1}{2}\left(1 + g\,x^2\right)^2 - 1$$

- ground state perturbatively zero (very convergent!)
- SUSY broken non-perturbatively (single-instanton)

$\mathcal{I}\bar{\mathcal{I}}$ interaction involves bosonic and fermionic zero modes

$$Z_1 = \frac{T}{\sqrt{\pi}}\,\frac{2}{\pi\,g^2}\int dt\,e^{-\frac{1}{3g^2}}\left(e^{\left(-2t+\frac{2}{g^2}\,e^{-2t}\right)} - 1\right)$$

Trans-series for Energy Eigenvalues

- perturbation theory: $E^{(N)}_{\text{pert. theory}}(g^2) = \sum_{k=0}^{\infty} g^{2k} E_k^{(N)}$
- non-Borel-summable: incomplete
- all non-perturbative multi-instanton terms:

"trans-series"

$$\begin{aligned} E^{(N)}(g^2) &= E^{(N)}_{\text{pert. theory}}(g^2) \\ &+ \sum_{k=1}^{\infty}\sum_{l=1}^{k-1}\sum_{p=0}^{\infty} \underbrace{\left(\frac{1}{g^{2N+1}}\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instanton}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-mode}} \underbrace{c_{k,l,p}\,g^{2p}}_{\text{perturbative fluctuations}} \end{aligned}$$

precisely of Écalle's trans-series form !

Decoding of Trans-series

$$f(g^2)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n}\left[\exp\left(-\frac{S}{g^2}\right)\right]^k\left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$

- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0}\, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0}\sim \alpha\frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i\,\pi\,\alpha\, e^{-2S/g^2}$

- but $c_{0,2,1}=-\alpha$: BZJ cancellation !

pert flucs about instanton: $e^{-S/g^2}\,\left(1+a_1 g^2+a_2 g^4+\dots\right)$

divergent:

$a_n\sim\frac{n!}{(2S)^n}\,(a\,\ln n+b)\Rightarrow \pm i\,\pi\, e^{-3S/g^2}\left(a\ln\frac{1}{g^2}+b\right)$

- 3-instanton: $e^{-3S/g^2}\left[\frac{a}{2}\left(\ln\left(-\frac{1}{g^2}\right)\right)^2+b\,\ln\left(-\frac{1}{g^2}\right)+c\right]$

resurgence: *ad infinitum*, also sub-leading large-order terms

Lecture 2

- divergence of perturbation theory in QFT
- Euler-Heisenberg effective actions
- curing the IR renormalon puzzle in $\mathbb{CP}^{N-1}$ models

- Hurst (1952): ϕ^4 perturbation theory is divergent:

(i) factorial growth of number of diagrams
(ii) explicit lower bounds on diagrams

> *If it be granted that the perturbation expansion does not lead to a convergent series in the coupling constant for all theories which can be renormalized, at least, then a reconciliation is needed between this and the excellent agreement found in electrodynamics between experimental results and low-order calculations. It is suggested that this agreement is due to the fact that the S-matrix expansion is to be interpreted as an asymptotic expansion in the fine-structure constant ...*
>
> *C. A. Hurst, 1952*

Dyson's argument (QED)

- Dyson (1952): *physical argument* for divergence of QED perturbation theory

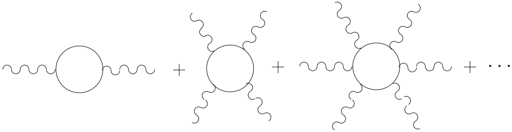
$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

> *Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*
>
> *F. J. Dyson, 1952*

- *suggests* perturbative expansion cannot be convergent

Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216



- 1-loop QED effective action in uniform background emag field
- e.g., constant B field:

$$S = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left(\coth s - \frac{1}{s} - \frac{s}{3} \right) \exp\left[-\frac{m^2 s}{eB} \right]$$

$$S = -\frac{e^2 B^2}{2\pi^2} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{2eB}{m^2} \right)^{2n+2}$$

Euler-Heisenberg Effective Action

- e.g., constant B field: characteristic factorial divergence

$$c_n = (-1)^{n+1}\,\frac{\Gamma(2n+2)}{8}\sum_{k=1}^{\infty}\frac{1}{(k\,\pi)^{2n+4}}$$

- recall Borel summation:

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\,\Gamma(\gamma\, n+\delta)$$

$$\rightarrow \quad f(g) \sim \frac{1}{\gamma}\int_0^{\infty}\frac{ds}{s}\left(\frac{1}{1+s}\right)\left(\frac{s}{\beta g}\right)^{\delta/\gamma}\exp\left[-\left(\frac{s}{\beta g}\right)^{1/\gamma}\right]$$

- reconstruct correct Borel transform:

$$\sum_{k=1}^{\infty}\frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

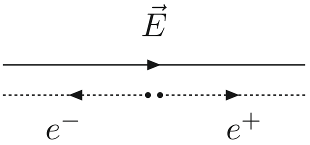
Euler-Heisenberg Effective Action

B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \mathrm{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$



Schwinger pair production from vacuum:
$\mathrm{Im}\, S \to$ physical pair production rate

- suggests Euler-Heisenberg series must be divergent

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even

- scalar QED Euler-Heisenberg in self-dual background ($F = \pm\tilde{F}$):

$$
\begin{aligned}
S &= \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right) \\
&= -\frac{m^4}{16\pi^2}\sum_{n=1}^\infty \frac{B_{2n+2}}{2n(2n+2)}\left(\frac{2F}{m^2}\right)^{2n+2}
\end{aligned}
$$

- "electric" self-dual ($F \to i\,F$):

$$
\operatorname{Im} S = \frac{m^2\, F}{32\pi^3}\sum_{k=1}^\infty \left(\frac{2\pi}{k} + \frac{2F}{k^2\, m^2}\right) e^{-\pi\, k\, m^2/F}
$$

- scalar QED EH in self-dual background ($F = \pm\tilde{F}$):

$$S = \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right)$$

- Gaussian matrix model: $\lambda = g\,N$

$$\mathcal{F} = -\frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right)$$

- $c = 1$ String: $\lambda = g\,N$

$$\mathcal{F} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sin^2(t)} - \frac{1}{s^2} - \frac{1}{3}\right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4}\sum_{m\in\mathbb{Z}}\int_0^\infty \frac{dt}{t}\, e^{-2(\lambda+2\pi\, i\, m)\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{s^2} + \frac{1}{3}\right)$$

• similar structure arises in more general topological string theories and matrix models

• resurgence and Borel-Écalle summation provide a natural framework for combining perturbative genus expansions with non-perturbative information

• Mariño, Schiappa, Pasquetti, Aniceto, Vonk, ...

key problem: analytic continuation of functional integrals

Resurgence and Analytic Continuation

one view (of many) of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

• "alien calculus" (Écalle)

• median resummation: encodes intricate combinatorics of cancellations in trans-series

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots + \frac{174611}{6600z^{20}} - \ldots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$

formal series $\quad \Rightarrow \quad \operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$

- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

reflection formula

$\Rightarrow \quad \operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$

"raw" asymptotics inconsistent with analytic continuation

- resurgence fixes this

$$\operatorname{Re} \psi(1+i\,y) \sim \ln y + 2 \sum_{n=0}^{\infty} \frac{(2n+1)!}{(2\pi y)^{2n+2}} \zeta(2n+2) - i\pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

• this example arises in many QFT and String Theory computations:

• Euler-Heisenberg, de Sitter, exact S-matrices, Chern-Simons partition functions, matrix models, ...

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\frac{n^2\,\pi^2}{L^2} - \lambda} &= -\frac{L^2}{2}\left(\frac{\cot\left(L\sqrt{\lambda}\right)}{L\sqrt{\lambda}} - \frac{1}{L^2\,\lambda}\right) \\
&= \frac{L}{2\pi\sqrt{\lambda}}\left(\psi\left(1+\frac{L\sqrt{\lambda}}{\pi}\right) - \psi\left(1-\frac{L\sqrt{\lambda}}{\pi}\right)\right)
\end{aligned}$$

Divergence of derivative expansion (GD, T. Hall, hep-th/9902064)

- time-dependent E field: $E(t) = E\,\mathrm{sech}^2\,(t/\tau)$

$$S = -\frac{m^4}{8\pi^{3/2}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(m\lambda)^{2j}}\sum_{k=2}^{\infty}(-1)^k\left(\frac{2E}{m^2}\right)^{2k}\frac{\Gamma(2k+j)\Gamma(2k+j-2)\mathcal{B}_{2k+2j}}{j!(2k)!\Gamma(2k+j+\frac{1}{2})}$$

- Borel sum perturbative expansion: large k (j fixed):

$$c_k^{(j)} \sim 2\frac{\Gamma(2k+3j-\frac{1}{2})}{(2\pi)^{2j+2k+2}}$$

$$\mathrm{Im}\,S^{(j)} \sim \exp\left[-\frac{m^2\pi}{E}\right]\frac{1}{j!}\left(\frac{m^4\pi}{4\tau^2E^3}\right)^j$$

- resum derivative expansion

$$\mathrm{Im}\,S \sim \exp\left[-\frac{m^2\pi}{E}\left(1-\frac{1}{4}\left(\frac{m}{E\tau}\right)^2\right)\right]$$

Divergence of derivative expansion

- Borel sum derivative expansion: large j (k fixed):

$$c_j^{(k)} \sim 2^{\frac{9}{2}-2k}\frac{\Gamma(2j+4k-\frac{5}{2})}{(2\pi)^{2j+2k}}$$

$$\mathrm{Im}\, S^{(k)} \sim \frac{(2\pi E\tau^2)^{2k}}{(2k)!}\, e^{-2\pi m\tau}$$

- resum perturbative expansion:

$$\mathrm{Im}\, S \sim \exp\left[-2\pi m\tau\left(1-\frac{E\tau}{m}\right)\right]$$

- compare:

$$\mathrm{Im}\, S \sim \exp\left[-\frac{m^2\pi}{E}\left(1-\frac{1}{4}\left(\frac{m}{E\tau}\right)^2\right)\right]$$

- different limits of full: $\mathrm{Im}\, S \sim \exp\left[-\frac{m^2\pi}{E} f\left(\frac{m}{E\tau}\right)\right]$
- derivative expansion must be divergent

Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

QFT: new physical effects occur, due to running of coupling constant with momentum

faster source of divergence: “renormalons”

$\Rightarrow$ leading non-perturbative exponentials

non-alternating factorial growth: $c_n \sim \beta^n \, \Gamma(\gamma\, n + \delta)$

positive Borel pole: $t_c = \left(\frac{1}{\beta\, g}\right)^{1/\gamma}$

non-perturbative exponential: $\pm i\, \frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$

UV and IR Renormalons

e.g. QED with N_f massless flavors

- Adler function $D(Q^2) = -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}$
- bubble-chains, momentum k $\rightarrow$ interpolating expression

$$D(Q^2) = Q^2 \int_0^\infty d(k^2) \frac{k^2\,\alpha_s(k^2)}{(k^2+Q^2)^3}$$

- running coupling $\alpha_s(k^2)$:

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 - \frac{\beta_0\,\alpha_s(Q^2)}{4\pi}\ln(Q^2/k^2)}$$

β_0: first beta-function coefficient

- expand $\alpha_s(k^2)$ in series in powers of $\alpha_s(Q^2)$:

$$D(Q^2) = \alpha_s(Q^2) \sum_{n=0}^{\infty} \left(\frac{\beta_0\,\alpha_s(Q^2)}{4\pi}\right)^n Q^2 \int_0^\infty d(k^2) \frac{k^2\,\left(\ln(Q^2/k^2)\right)^n}{(k^2+Q^2)^3}$$

UV and IR Renormalons

$$D(Q^2) = \alpha_s(Q^2) \sum_{n=0}^{\infty} \left(\frac{\beta_0\, \alpha_s(Q^2)}{4\pi} \right)^n Q^2 \int_0^\infty d(k^2) \frac{k^2\ \left(\ln(Q^2/k^2)\right)^n}{(k^2+Q^2)^3}$$

- IR low momentum: split at $k^2 = Q^2$ $(y \equiv 2\ln(Q^2/k^2))$

$$\begin{aligned} Q^2 \int_0^{Q^2} d(k^2) \frac{k^2\ \left(\ln(Q^2/k^2)\right)^n}{(k^2+Q^2)^3} &= \frac{1}{2}\frac{1}{2^n} \int_0^\infty dy \frac{e^{-y}\, y^n}{\left(1+e^{-y/2}\right)^3} \\ &\sim \frac{n!}{2} \left(\frac{1}{2^n} - \frac{2}{3^n} + O\left(\frac{1}{4^n}\right)\right) \end{aligned}$$

- UV high momentum: $(\bar{y} \equiv \ln(k^2/Q^2))$

$$\begin{aligned} Q^2 \int_{Q^2}^{\infty} d(k^2) \frac{k^2\ \left(\ln(Q^2/k^2)\right)^n}{(k^2+Q^2)^3} &= (-1)^n \int_0^\infty d\bar{y} \frac{e^{-\bar{y}}\, \bar{y}^n}{\left(1+e^{-\bar{y}}\right)^3} \\ &\sim (-1)^n n! \left(1 - \frac{3}{2^{n+1}} + O\left(\frac{1}{3^n}\right)\right) \end{aligned}$$

UV and IR Renormalons

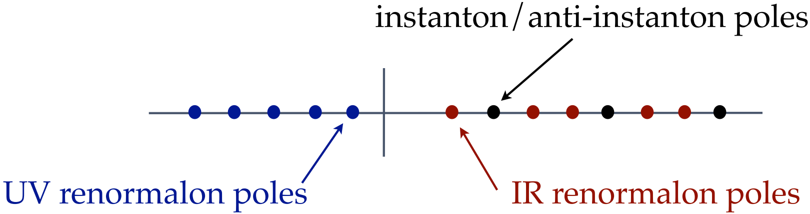
renormalon poles:

$$t_n^{IR} = +\frac{4\pi}{\beta_0}\, n \quad , \quad n = 2, 3, 4, \ldots$$

$$t_n^{UV} = -\frac{4\pi}{\beta_0}\, n \quad , \quad n = 1, 2, 3, \ldots$$

Borel poles due to renormalons are closer to the origin:

"dominant effect"



IR Renormalon Puzzle in Asymptotically Free QFT

Asymptotically free QFT's: e.g. Yang-Mills or $\mathbb{CP}^{N-1}$

(i) degenerate classical vacua

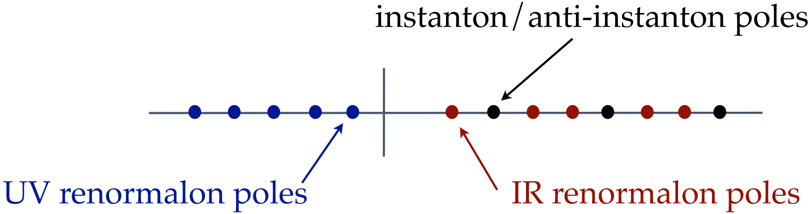
(ii) non Borel summable perturbation theory, due to infrared (IR) renormalons, associated with IR-momentum behaviour of certain bubble-chain diagrams

- IR renormalons $\Rightarrow$ perturbation theory is ill-defined and incomplete
- for $\mathbb{CP}^{N-1}$ on $\mathbb{R}^2$, or YM on $\mathbb{R}^4$, BZJ mechanism does not work, because Borel poles are in 'wrong' locations

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow$ $\pm i\, e^{-2S/\beta_0}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow$ $\pm i\, e^{-2S}$



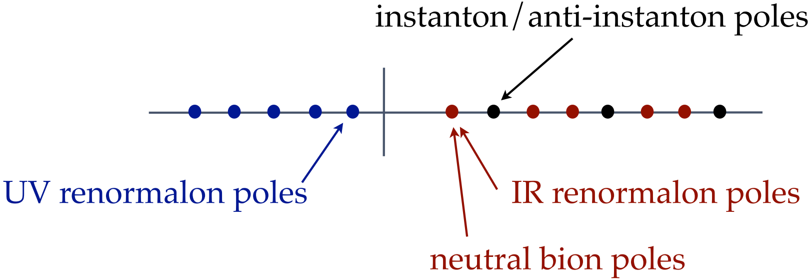
appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, GD, Ünsal, 1206.1890 1210.2423)

- scale modulus of instantons
- spatial compactification and principle of continuity

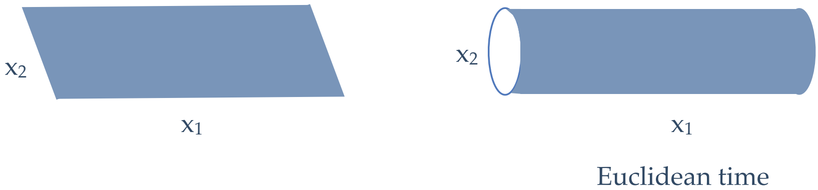


cancellation occurs !

Topological Molecules in Spatially Compactified Theories

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification

$$\mathbb{R}^2 \to S_L^1 \times \mathbb{R}^1$$



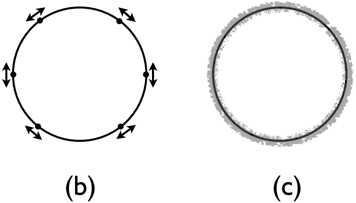
instantons fractionalize

temporal conpactification: information only about deconfined phase

spatial compactification: semi-classical small L regime continuously connected to large L:

principle of continuity

Weak Coupling Non-Trivial Holonomy and Center Symmetry



- (a) **Weak coupling trivial holonomy:** Semi-classical OK, but disconnected from strong coupling regime
- (b) **Weak coupling non-trivial holonomy:** Semi-classical analysis OK, and continuously connected to strong coupling regime
- (c) **Strong coupling non-trivial holonomy:** Weak-coupling semi-classical analysis not OK

- monopole-instantons, $\mathcal{M}_i$, or kink-instantons $\mathcal{K}_i$, $i = 1, 2, \ldots, N$.
- Charged bions (correlated kink-anti-kink events): $\mathcal{B}_{ij} = [\mathcal{M}_i \bar{\mathcal{M}}_j]$, or $\mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j]$, with $i \neq j$
- Neutral bions: $\mathcal{B}_{ii} = [\mathcal{M}_i \bar{\mathcal{M}}_i]$, and $\mathcal{B}_{ii} = [\mathcal{K}_i \bar{\mathcal{K}}_i]$
- Neutral bion-anti-bion molecular events such as $[\mathcal{B}_{ij}\mathcal{B}_{ji}]$, $[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]$, etc ...

hierarchy of scales:

$$\begin{array}{ccccccc}
r_{\rm k} & \ll & r_{\rm b} & \ll & d_{\rm k-k} & \ll & d_{\rm b-b}, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \ll & L \log\left(\frac{1}{g^2}\right) & \ll & L\, e^{S_0} & \ll & L\, e^{2S_0}.
\end{array}$$

Graded Resurgence Triangle

- Perturbation theory is independent of topological Θ-angle $\Rightarrow$ ambiguity due to non-Borel summability of perturbation theory is also independent of Θ.
- $\Rightarrow$ non-Borel summability of large orders of perturbation theory can *never* be cancelled by non-perturbative configurations with non-vanishing topological charge. Can *only* be cancelled by topological configurations with zero topological charge, or equivalently, without any Θ-angle dependence

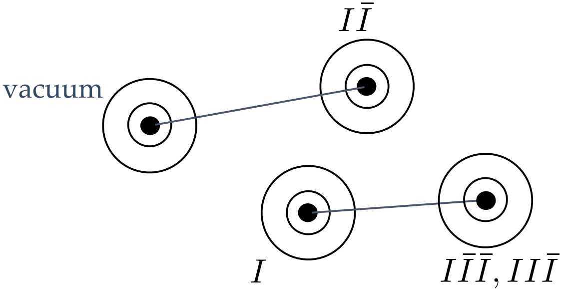
Graded Resurgence Triangle

saddle points labelled by: $[n, m]$

$$n = n_{instanton} + n_{anti-instanton} \quad , \quad m = n_{instanton} - n_{anti-instanton}$$

$$\begin{array}{ccccccccc}
 & & & & [0,0] & & & & \\
 & & & [1,1] & & [1,-1] & & & \\
 & & [2,2] & & [2,0] & & [2,-2] & & \\
 & [3,3] & & [3,1] & & [3,-1] & & [3,-3] & \\
[4,4] & & [4,2] & & [4,0] & & [4,-2] & & [4,-4] \\
\ddots & & & & \vdots & & & & \ddots
\end{array}$$

Graded Resurgence Triangle



$$(1 + a_1 g^2 + a_2 g^4 + \cdots + \ldots) \quad + e^{-S/g^2}(1 + b_1 g^2 + b_2 g^4 + \cdots + \ldots)$$

$$+ e^{-2S/g^2}(1 + c_1 g^2 + c_2 g^4 + \cdots + \ldots) + e^{-3S/g^2}(1 + d_1 g^2 + d_2 g^4 + \cdots + \ldots)$$

$$+ \ldots \qquad + (\text{log terms})$$

$\mathbb{CP}^{N-1}$ Model

$\mathbb{CP}^{N-1}$ model: two-dim. sigma model analog of Yang-Mills

- asymptotically free: $\beta_0 = N$ (independent of N_f)
- instantons, theta vacua, fermion zero modes, ...
- divergent perturbation theory (non-Borel summable)
- renormalons (both UV and IR)
- large-N analysis
- non-perturbative mass gap: $m_g = \Lambda = \mu\, e^{-4\pi/(g^2 N)}$
- couple to fermions, SUSY, ...
- 'unstable' finite action non-self-dual classical solutions (path integral saddle points)

Basics of $\mathbb{CP}^{N-1}$ Model

- classical bosonic action: $S = \frac{2}{g^2} \int d^2x \, (D_\mu n)^\dagger \, D_\mu n$
- $n = N$-component column vector with $n^\dagger n = 1$
- local $U(1)$ symmetry , global $U(N)$ symmetry
- $D_\mu = \partial_\mu + i A_\mu$, with abelian gauge field $A_\mu = i \, n^\dagger \partial_\mu n$
- target space $\mathcal{M}_{N,1} \equiv \mathbb{CP}^{N-1} = \frac{U(N)}{U(N-1) \times U(1)}$
- $N^2 - 1 - (N-1)^2 = 2(N-1)$ real fields
- topological charge

$$Q = -\frac{i}{2\pi} \int d^2x \, \epsilon_{\mu\nu} \partial_\mu \left(n^\dagger \partial_\nu n \right) = \frac{1}{2\pi} \int d^2x \, \epsilon_{\mu\nu} \partial_\mu A_\nu$$

- couple to fermions

$$S_{\text{fermion}} = \frac{2}{g^2} \int \left[-i \bar{\psi} \gamma_\mu D_\mu \psi + \frac{1}{4} \left((\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_3\psi)^2 - (\bar{\psi}\gamma_\mu\psi)^2 \right) \right]$$

Bogomolny factorization:

$$(D_\mu n)^\dagger D_\mu n = |(D_\mu \pm i\epsilon_{\mu\nu} D_\nu)\, n|^2 \mp i\, \epsilon_{\mu\nu} \partial_\mu \left(n^\dagger \partial_\nu n \right)$$

self-dual instanton equations

$$D_\mu n = \mp i\, \epsilon_{\mu\nu} D_\nu n$$

homogeneous fields: $n \equiv \frac{v}{|v|}$

$$\text{instanton} : v = v(z) \quad , \quad \text{anti-instanton:} \, v = v(\bar{z})$$

e.g., simplest instanton for $\mathbb{CP}^1$ on $\mathbb{R}^2$:

$$v = \begin{pmatrix} 1 \\ (z-b)/a \end{pmatrix} \quad \Rightarrow \quad Q = \frac{1}{\pi} \int d^2x \, \frac{|a|^2}{(|a|^2 + |z-b|^2)^2} = 1$$

$2(N-1)$ angular fields:

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos\frac{\theta_1}{2} \\ e^{i\varphi_2} \sin\frac{\theta_1}{2}\, \cos\frac{\theta_2}{2} \\ e^{i\varphi_3} \sin\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2}\, \cos\frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2}\, \sin\frac{\theta_3}{2}\, \ldots \sin\frac{\theta_{N-1}}{2} \end{pmatrix}$$

order parameter:

$$\Omega(x_1) = \begin{pmatrix} e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]} & 0 & \ldots & 0 \\ 0 & e^{i[\varphi_2(x_1,0)-\varphi_2(x_1,L)]} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & e^{i[\varphi_N(x_1,0)-\varphi_N(x_1,L)]} \end{pmatrix}$$

twisted b.c.'s: $\mathbb{Z}_N \; : \; \Omega \longrightarrow e^{i\frac{2\pi k}{N}} \; \Omega$

“Center Symmetry” in $\mathbb{CP}^{N-1}$

$$V_-[\Omega] = \frac{2}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1+(-1)^n N_f)(|\mathrm{tr}\,\Omega^n|-1) \qquad \text{(thermal)}$$

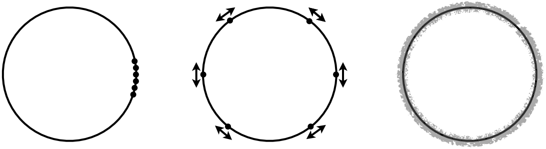
$$V_+[\Omega] = (N_f-1)\frac{2}{\pi L^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(|\mathrm{tr}\,\Omega^n|-1) \qquad \text{(spatial)}$$

minima:

$$\Omega_0^{\text{thermal}} = e^{i\frac{2\pi k}{N}}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad \text{(thermal)}$$

$$\Omega_0^{\text{spatial}} = \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(N-1)}{N}} \end{pmatrix} \qquad \text{(spatial)}$$

- $\mathbf{N_f > 1}$: repulsive interaction between eigenvalues of holonomy Ω: center symmetry preserved
- $\mathbf{N_f = 1}$: $\mathcal{N} = (2,2)$ SUSY $\mathbb{CP}^{N-1}$: perturbative potential vanishes to all orders (SUSY). Non-perturbatively induced potential stabilizes center-symmetry
- $\mathbf{N_f = 0}$: deformed $\mathbb{CP}^{N-1}$, or integrating out heavy fermions



(a) (b) (c)

Fractionalized Instantons in $\mathbb{CP}^{N-1}$

- untwisted instanton on $\mathbb{R}^1 \times \mathbb{S}^1_L$:

$$v = \begin{pmatrix} 1 \\ \lambda_1 + \lambda_2 \, e^{-\frac{2\pi}{L} z} \end{pmatrix}$$

- *spatial* twist:

$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2 \, e^{-\frac{2\pi}{L} z}\right) \, e^{\frac{2\pi}{L} \mu_2 \, z} \end{pmatrix}$$

- twisted boundary condition $\Rightarrow$ factor $e^{\frac{2\pi i}{L} \mu_2 \, x_2}$
- but, non-holomorphic: $\Rightarrow$ factor $e^{\frac{2\pi}{L} \mu_2 \, z}$

$\Rightarrow$ twist in x_2 also prescribes dependence in non-compact direction x_1

Fractionalized Instantons in $\mathbb{CP}^{N-1}$

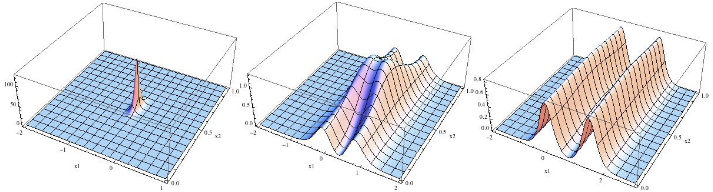


Figure: $Q = 1$ instanton in $\mathbb{CP}^1$, $(N = 2)$, in weak coupling center-symmetric background. Small circle: instanton splits into two $Q = \frac{1}{2}$ instantons.

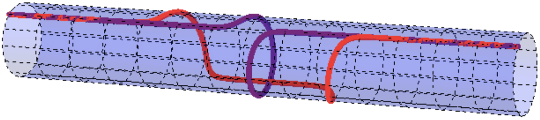


Figure: Wilson loop for small $Q = 1$ instanton (purple). Large instanton (red) splits into two separate kink-instantons. Each wraps half-way around the cylinder.

Fractionalized Instantons in $\mathbb{CP}^{N-1}$

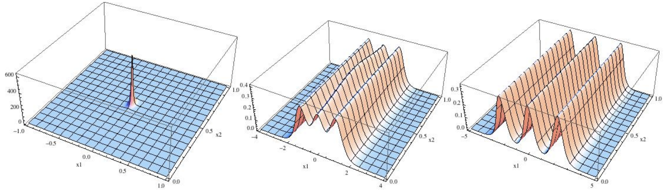


Figure: $Q = 1$ instanton in $\mathbb{CP}^2$, $(N = 3)$, in weak coupling center-symmetric background. Small circle: instanton splits into three $Q = \frac{1}{3}$ instantons.

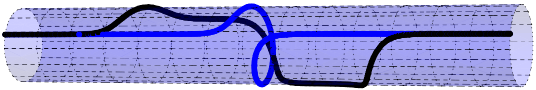


Figure: Wilson loop for small $Q = 1$ instanton (blue). Large instanton (black) splits into three separate kink-instantons. Each wraps one-third-way around the cylinder.

Fractionalized Instantons in $\mathbb{C}\mathbb{P}^{N-1}$

- fundamental fractionalized instantons with $Q = \frac{1}{N}$
- bosonic zero modes:

$$2N \xrightarrow{\text{short-distance}} 2 + 1 + (2N-3) = (\mathbf{a}_I \in \mathbb{R}^2) + (\rho \in \mathbb{R}^+) + (\text{orient.})$$

$$2N \xrightarrow{\text{long-distance}} N[1+1] = N[(a \in \mathbb{R}) + (\phi \in U(1))]$$

Fractionalized Bions in $\mathbb{CP}^{N-1}$

- bions: topological molecules of instantons/anti-instantons
- characterized by (extended) Cartan matrix (as in YM)
- "orientation" dependence of $\mathcal{I}\bar{\mathcal{I}}$ interaction:
- charged bions: $\hat{A}_{ij} < 0$; repulsive bosonic interaction

$$\mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j] \sim e^{-S_i(\varphi)-S_j(\varphi)} e^{i\sigma(\alpha_i-\alpha_j)}$$

- neutral bions: $\hat{A}_{ii} > 0$; attractive bosonic interaction

$$\Re \mathcal{B}_{ii} = \Re[\mathcal{K}_i \bar{\mathcal{K}}_i] \sim e^{-2S_i(\varphi)}$$

Fractionalized Bions in $\mathbb{CP}^{N-1}$

- charged bions:

$$\mathcal{A}_{ij} = \mathcal{A}_i \mathcal{A}_j \left(\frac{\alpha_i . \alpha_j}{2}\right)^{2N_f} \left(\frac{g^2}{2L}\right)^{2N_f} 2 \int_0^\infty d\tau \, e^{-V_{\text{eff}}^{ij}(\tau)}$$

where $(\xi \equiv \frac{2\pi}{N\,L})$

$$V_{\text{eff}}^{ij}(\tau) = -8\xi \frac{\alpha_i . \alpha_j}{g^2} e^{-\xi\tau} + 2N_f \xi \tau$$

- characteristic scale dominating the integral:

$$\tau^* = \frac{1}{\xi} \log\left(\frac{4\pi}{g^2 N N_f}\right) \quad , \quad r_{\text{b}} = r_{\text{k}} \log\left(\frac{4\pi}{g^2 N N_f}\right) \qquad N_f \geq 1$$

- quasi-zero mode integral:

$$I(g^2) \; = \int_0^\infty d\tau \exp\left[-\left(\frac{4\xi}{g^2} e^{-\xi\tau} + 2N_f \xi\tau\right)\right] = \left(\frac{g^2}{4\xi}\right)^{2N_f} \int_0^{\frac{4\xi}{g^2}} du \; e^{-u} \; u^{2N_f - 1}$$

$$\underbrace{\longrightarrow}_{g^2 \ll 1} \left(\frac{g^2}{4\xi}\right)^{2N_f} \Gamma(2N_f) = \left(\frac{g^2 N}{8\pi}\right)^{2N_f} \Gamma(2N_f)$$

Fractionalized Bions in $\mathbb{CP}^{N-1}$

- neutral bions:

$$\widetilde{I}(g^2) = \int_0^\infty d\tau \exp\left[-\left(-\frac{8\xi}{g^2}e^{-\xi\tau} + 2N_f\xi\tau\right)\right]$$

- both bosonic and fermionic zero mode induced interactions are attractive (as in gauge theory)

- semi-classical $[\mathcal{K}_i\bar{\mathcal{K}}_i]$ configuration seems meaningless

N.B. $[\mathcal{K}_i\bar{\mathcal{K}}_i]$ has same quantum nos. as pert. vacuum

- generalized BZJ-prescription: deform the contour of integration, or equivalently, rotate $g^2 \to g^2 e^{i\theta}$

$$\tilde{I}(g^2, N_f) \to I(-g^2, N_f) = \left(-\frac{g^2 N}{8\pi}\right)^{2N_f} \Gamma(2N_f)$$

- $N_f = 0$: ambiguous result:

$$\tilde{I}(g^2, N_f = 0) = \left(\log\left(-\frac{g^2 N}{8\pi}\right) - \gamma\right) = I(g^2) \pm i\pi$$

- neutral bions: same ambiguity as in bosonic QM (Bogomolny)
- kink-anti-kink amplitude is two-fold ambiguous:

$$[\mathcal{K}_i\bar{\mathcal{K}}_i]_{\theta=0^\pm} = \left(\log\left(\frac{g^2N}{8\pi}\right) - \gamma\right) 2\mathcal{A}_i^2 e^{-2S_0} \pm i\pi 2\mathcal{A}_i^2 e^{-2S_0}$$

Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- small radius limit: effective QM Hamiltonian

$$H^{\text{zero}}_{\alpha_k} = \frac{g^2}{2} P_\theta^2 + \frac{\xi^2}{2g^2} \sin^2\theta + \frac{g^2}{2\sin^2\theta} P_\phi^2, \qquad \xi = \frac{2\pi}{N}, \qquad (\text{set } L = 1)$$

- Born-Oppenheimer approximation: drop high ϕ-sector modes

effective Mathieu equation:

$$\psi'' + \left(p + \frac{\xi^2}{2g^2} \cos(2g\theta) \right) \psi = 0, \qquad p = 2E - \frac{\xi^2}{2g^2}$$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \quad a_q \sim -\frac{2}{\pi} \left(\frac{1}{4\xi} \right)^q q! \left(1 - \frac{5}{2q} + O(q^{-2}) \right)$$

- non-Borel summable!

Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \quad a_q \sim -\frac{2}{\pi} \left(\frac{1}{4\xi}\right)^q q! \left(1 - \frac{5}{2q} + O(q^{-2})\right)$$

- lateral Borel summation $\Rightarrow$

$$\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_\pm} dt\ B\mathcal{E}(t)\ e^{-t/g^2} = \Re \mathcal{S}\mathcal{E}(g^2) \mp i \frac{8\xi}{g^2} e^{-\frac{4\xi}{g^2}}$$
$$= \Re \mathbb{B}_0 \mp i \frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$

- perturbative sector: lateral Borel-Écalle summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

- non-perturbative sector: bion-bion amplitudes

$$\left[\mathcal{K}_i\bar{\mathcal{K}}_i\right]_\pm = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact cancellation !

application of resurgence to nontrivial QFT

Graded Resurgence Triangle and Extended SUSY

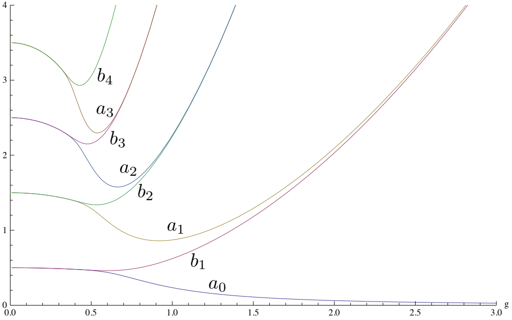
extended SUSY: no superpotential; no bions; no condensates

$$
\begin{array}{ccccccccc}
 & & & & [0,0] & & & & \\
 & & & [1,1] & & [1,-1] & & & \\
 & & [2,2] & & \varnothing & & [2,-2] & & \\
 & [3,3] & & \varnothing & & \varnothing & & [3,-3] & \\
[4,4] & & \varnothing & & \varnothing & & \varnothing & & [4,-4] \\
\iddots & & & & \vdots & & & & \ddots
\end{array}
$$

no cancellations can occur

$\Rightarrow$ perturbative expansions must be Borel summable !

Microsopic Origin of Mass Gap in Twisted $\mathbb{CP}^{N-1}$



$$h = \frac{\xi}{2g^2} = \frac{2\pi}{N(2g^2)}$$

$$\begin{aligned}
\Delta E_n &= \frac{g^2}{2}\left(b_{n+1}\left(h^2\right) - a_n\left(h^2\right)\right) \\
&= \frac{g^2}{2}\left(\frac{2^{4n+5}}{n!}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} h^{n+\frac{3}{2}} e^{-4h}\left(1 - \frac{6n^2+14n+7}{32h} + O\left(\frac{1}{h^2}\right)\right)\right)
\end{aligned}$$

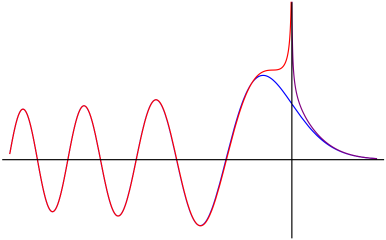
$$m_g = \frac{C}{N\,g^2}\left(1 - \frac{7\,N\,g^2}{32\pi} + O\left((N\,g^2)^2\right)\right) e^{-\frac{4\pi}{N\,g^2}} \sim e^{-S_I/N}$$

Lecture 3

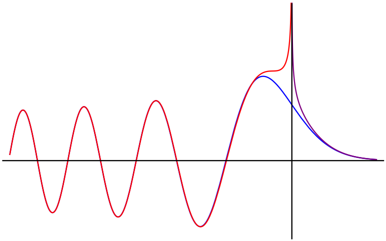
- Stokes Phenomenon
- Uniform WKB and origin of trans-series structure
- all non-perturbative orders from perturbation theory

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]



- supernumerary rainbows



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3+x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x\to+\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x\to-\infty \end{cases}$$

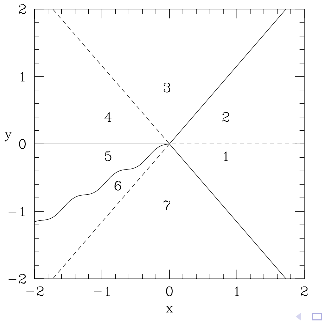
The Stokes Phenomenon

Stokes: • how to reconcile two exponentials in one region with one exponential in another region?

• how to reconstruct an analytic solution from non-analytic approximations?

Stokes line: real exponentials

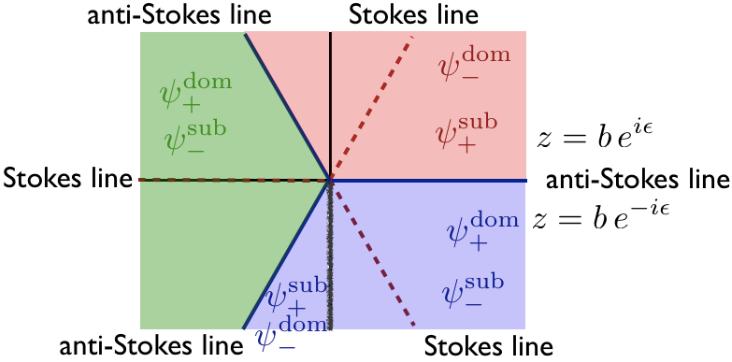
anti-Stokes line: imaginary exponentials



$$\hbar^2 \psi'' + Q^2 \psi = 0$$

Stokes phenomenon: suppose Q has a simple zero at z=0

$$\psi_\pm \sim \frac{e^{\pm i z^{3/2}}}{z^{1/4}} \qquad \psi_\pm \sim e^{\pm i b^{3/2}(\cos(\frac{3\epsilon}{2}) + i \sin(\frac{3\epsilon}{2}))}$$



The Stokes Phenomenon

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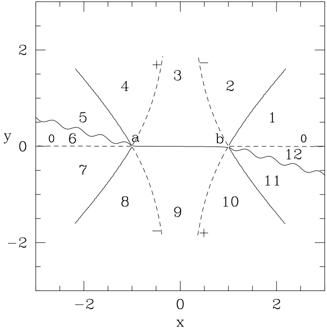
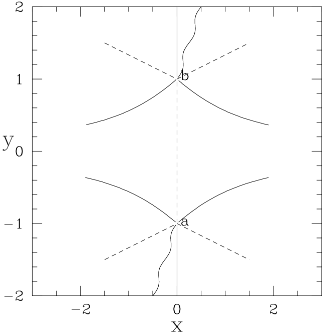




The Stokes Phenomenon

Stokes:
- how to reconcile two exponentials in one region with one exponential in another region?
- how to reconstruct an analytic solution from non-analytic approximations?

different exponentials turn on/off crossing between sectors

- universal smooth behavior (Stokes, Berry)

> *The inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficients changed. The range during which the inferior term remains in a mist decreases indefinitely as the [asymptotic parameter] increases indefinitely.*
>
> *G. G. Stokes, 1902*

- intricate monodromy behaviour

Universal large-order WKB

- Liouville-Green (WKB) approximation:

$$\hbar^2\,\psi'' + Q(x)\,\psi(x) = 0 \quad \to \quad \psi_\pm(x) \sim \frac{e^{\pm i/\hbar \int^x \sqrt{Q}}}{Q^{1/4}}\,(1 + \dots)$$

- all-orders expansion: Dingle's universal large-order form:

$$\psi_\pm(S) \sim \frac{1}{\sqrt{S'}}\, e^{\pm i\, S/\hbar} \sum_{n=0}^{\infty} n! \left(\frac{\pm i\,\hbar}{2\,S}\right)^n$$

- "singulant" variable: $S = \int^x \sqrt{Q}$

- exponential asymptotics of special functions, and of wavefunctions

Resurgence and Analytic Continuation

resurgence can be viewed as a method for making asymptotic expansions consistent with global analytic continuation properties

e.g.: asymptotics of special functions

- zero-dimensional partition functions

$$
\begin{aligned}
Z_1(\lambda) &= \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2\lambda}\sinh^2(\sqrt{\lambda}\,x)} = \frac{1}{\sqrt{\lambda}}\, e^{\frac{1}{4\lambda}}\, K_0\left(\frac{1}{4\lambda}\right) \\
&\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^n (2\lambda)^n \frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{Borel-summable}
\end{aligned}
$$

$$
\begin{aligned}
Z_2(\lambda) &= \int_{0}^{\pi/\sqrt{\lambda}} dx\, e^{-\frac{1}{2\lambda}\sin^2(\sqrt{\lambda}\,x)} = \frac{\pi}{\sqrt{\lambda}}\, e^{-\frac{1}{4\lambda}}\, I_0\left(\frac{1}{4\lambda}\right) \\
&\sim \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (2\lambda)^n \frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{non-Borel-summable}
\end{aligned}
$$

- connection formula: $K_0(e^{\pm i\pi}\,|z|) = K_0(|z|) \mp i\,\pi\, I_0(|z|)$

- Borel summation

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)$$

- lateral Borel summation

$$\begin{aligned}
&Z_1(e^{i\pi}\,\lambda) - Z_1(e^{-i\pi}\,\lambda)\\
&= \sqrt{\frac{\pi}{2}}\frac{1}{2\lambda}\int_1^\infty dt\, e^{-\frac{t}{2\lambda}}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t-i\varepsilon\right) - {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t+i\varepsilon\right)\right]\\
&\quad = \quad -(2i)\sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\,e^{-\frac{1}{2\lambda}}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)\\
&\quad = \quad -2\,i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)
\end{aligned}$$

- connection formula: $Z_1(e^{\pm i\pi}\,\lambda) = Z_2(\lambda) \mp i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)$

what changes going from linear to nonlinear ODE's ?

• Painlevé functions are generalization of special functions to nonlinear ODE's: many physical applications: statistical physics, optics, QFT, strings, ...

• resurgent trans-series are the natural language for their asymptotics

see: Mariño, Schiappa, Aniceto, Pasquetti, Vonk, ...

Resurgence in Nonlinear ODEs

- physical example: Painlevé I:

(i) all-genus solution of c=0 2d gravity

(ii) double-scaling limit of quartic matrix model

- perturbative amplitudes generated by series solution of Painlevé I: $\mathcal{F}''(z) = u(z)$, where

$$u^2(z) - \frac{1}{6}u'' = z$$

- non-perturbative results (Shenker, David, ...):

$$\mathcal{F}^{(1)}(z) = \frac{i}{8\sqrt{\pi}\,3^{3/4}\,z^{5/8}} \exp\left[-\frac{8\sqrt{3}}{5}\,z^{5/4}\right]\left(1 - \frac{37}{64\sqrt{3}}\frac{1}{z^{5/4}} + \dots\right)$$

- resurgence framework ...

Resurgence in Nonlinear ODEs: e.g. Painlevé I

Painlevé I: $u^2(z) - \frac{1}{6}u'' = z$

- $u = \sqrt{z}\, w(z)$
- $\xi = z^\alpha$, matching powers $\Rightarrow \alpha = \frac{5}{4}$:

$$\frac{d^2 w}{d\xi^2} + \frac{1}{\xi}\frac{dw}{d\xi} - \frac{4}{25}\frac{w}{\xi^2} = \frac{96}{25}(w^2 - 1)$$

- ansatz: $w \sim \frac{e^{-\xi}}{\sqrt{\xi}} \sum_{n=0}^\infty \frac{a_n^{(0)}}{\xi^{2n}} \quad \Rightarrow a^{(0)} = \left\{1, -\frac{1}{48}, -\frac{49}{4608}, \dots\right\}$
- large-order behavior non-Borel-summable:

$$a_n^{(0)} \sim -\Gamma\left(2n - \frac{1}{2}\right)\left(\frac{8\sqrt{3}}{5}\right)^{-2n+1/2} \frac{3^{1/4}}{2\,\pi^{3/2}}\left(1 + O\left(\frac{1}{n}\right)\right)$$

- imaginary part: $\mathrm{Im} \sim \pm \frac{e^{-A\,\xi}}{\sqrt{\xi}} \quad , \quad A \equiv \frac{8\sqrt{3}}{5}$

Resurgence in Nonlinear ODEs: e.g. Painlevé I

$$\text{Painlevé I:} \qquad \frac{d^2w}{d\xi^2} + \frac{1}{\xi}\frac{dw}{d\xi} - \frac{4}{25}\frac{w}{\xi^2} = \frac{96}{25}(w^2 - 1)$$

- perturbative ansatz: $w \sim \frac{e^{-\xi}}{\sqrt{\xi}} \sum_{n=0}^{\infty} \frac{a_n^{(0)}}{\xi^{2n}}$
- non-perturbative term: $\text{Im} \sim \pm \frac{e^{-A\,\xi}}{\sqrt{\xi}}$, $\quad A \equiv \frac{8\sqrt{3}}{5}$
- nonlinearity $\Rightarrow$ also need $e^{\mp l\,A\,\xi}$ terms, $l \in \mathbb{Z}^+$
- *double trans-series ansatz*:

$$w \sim \frac{e^{-\xi}}{\sqrt{\xi}} \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \sigma_1^l\, \sigma_2^k\, e^{-(l-k)A\,\xi}\, \mathcal{F}_{(l,k)}\left(\frac{1}{\xi}\right)$$

- resurgence $\Rightarrow$ $\quad \mathcal{F}_{(l,k)}\left(\frac{1}{\xi}\right)$ fluctuations entwined
- full resurgent details still being investigated (sectors and analytic continuation)

Resurgence in Nonlinear ODEs: e.g. Painlevé II

Painlevé II:

$$u'' - 2u^3(z) + 2\,z\,u(z) = 0$$

perturbative solution is non-Borel-summable

$\Rightarrow$ trans-series solution(s)

- Tracy-Widom law for statistics of max. eigenvalue for Gaussian random matrices
- double-scaling limit in 2d Yang-Mills
- double-scaling limit in unitary matrix models
- all-genus solution of 2d supergravity

Uniform WKB and Resurgent Trans-Series for Eigenvalues

- origin of trans-series structure (GD, Ünsal, 1306.4405, 1401.5202)

$$-g^4 \frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\, E\, \psi(y)$$

where

$$V_{\rm DW}(y) = y^2(1+y)^2 \quad , \quad V_{\rm SG}(y) = \sin^2(y)$$

- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \qquad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

$$\frac{d^2\psi}{dx^2} + \frac{p^2(x)}{\hbar^2}\,\psi(x) = 0$$

- uniform WKB: "comparison functions"

uniform approxs. are smooth at turning points ($p = 0$)

$$\psi = \frac{1}{\sqrt{S'(x)}}\,\phi(S(x)) \quad \Rightarrow \quad \frac{d^2\phi}{dS^2} + \frac{P^2(S)}{\hbar^2}\,\phi(S) = 0$$

- $P^2(S) = \text{constant} \to$ usual WKB: $\psi(x) = \frac{e^{i\int p}}{\sqrt{p(x)}}$
- $P^2(S) = S \to$ uniform $\psi(x) = \frac{\left(\int^x p\right)^{1/6}}{\sqrt{p(x)}}\,Ai\left(\frac{3}{2}\left(\int^x p\right)^{2/3}\right)$
- $P^2(S) = S^2 \to$ uniform $\psi = \frac{\left(\int^x p\right)^{1/4}}{\sqrt{p(x)}}\,D_{\frac{(E-1)}{2}}\left(\left(2\int^x p\right)^{1/2}\right)$

Uniform WKB and Resurgent Trans-Series for Eigenvalues

- uniform WKB ansatz (ν a parameter)

$$\psi(y) = \frac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$$

- nonlinear equation for $u(y)$:

$$V(y) - \frac{1}{4}u^2(u')^2 - g^2\,E + g^2\left(\nu + \frac{1}{2}\right)(u')^2 + \frac{g^4}{2}\sqrt{u'}\left(\frac{u''}{(u')^{3/2}}\right)' = 0$$

- perturbative expansion $\rightarrow u(y)$ and energy:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k}E_k(\nu)$$

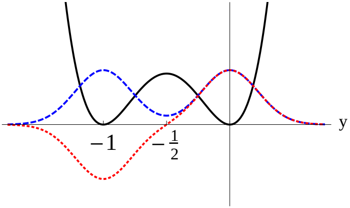
- $\nu = N$: Rayleigh-Schrödinger perturbation theory:

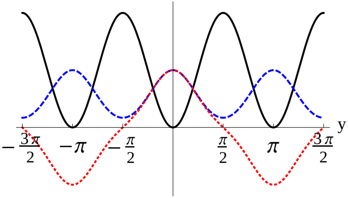
$$E\left(\nu = N, g^2\right) \equiv E^{(N)}_{\text{pert. theory}}(g^2)$$

- not Borel summable !

Uniform WKB and Resurgent Trans-Series for Eigenvalues

- global analysis $\Rightarrow$ boundary conditions:





- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow \quad g^2 \to e^{\pm i\,\epsilon}\, g^2$

$$D_\nu(z) \sim z^\nu \, e^{-z^2/4} \left(1 + \dots\right) + e^{\pm i\pi\nu} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} z^{-1-\nu} \, e^{z^2/4} \left(1 + \dots\right)$$

$\longrightarrow \quad$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)} \left(\frac{e^{\pm i\pi}\, 2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}} \, \mathcal{F}(\nu, g^2)$$

- exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu}=\frac{e^{-S/g^2}}{\sqrt{\pi\,g^2}}\,\mathcal{F}(\nu,g^2)$$

- expand $\nu = N+\delta\nu$:

$$LHS=-N!\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-N}\left\{\delta\nu-\left[\gamma+\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)-h_N\right](\delta\nu)^2+\dots\right\}$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi\equiv\frac{e^{-S/g^2}}{\sqrt{\pi\,g^2}}$):

$$\begin{aligned}\nu &= N+\frac{\left(\frac{2}{g^2}\right)^N\mathcal{F}(N,g^2)}{N!}\,\xi\\ &-\frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{F}\frac{\partial\mathcal{F}}{\partial N}+\left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)-\psi(N+1)\right)\mathcal{F}^2\right]\xi^2+O(\xi^3)\end{aligned}$$

- insert: $E=E(\nu,g^2)=\sum_{k=0}^{\infty}g^{2k}E_k(\nu)$ $\Rightarrow$ trans-series!

conclusion:

for QM problems with degenerate harmonic vacua, the trans-series form of the exact expressions for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions

generic and universal

Zinn-Justin/Jentschura conjecture: generate entire trans-series from

(i) perturbative expansion $E = E(\nu, g^2) \quad (\nu = \nu(E, g^2))$
(ii) single-instanton fluctuation function $\mathcal{F}(E, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

uniform WKB approach explains why this is the case

Connecting Perturbative/Non-Perturbative Sectors (GD,Unsal, 1401.5202)

Zinn-Justin/Jentschura: $\mathcal{F}(E,g) \sim \exp[-A(E,g)/2]$

- perturbative function: $(B \equiv \nu + \frac{1}{2})$

$$B_{\rm DW}(E,g) = E + g\left(3E^2+\frac{1}{4}\right) + g^2\left(35E^3+\frac{25}{4}E\right) + g^3\left(\frac{1155}{2}E^4+\frac{735}{4}E^2+\frac{175}{32}\right)+\dots$$

- non-perturbative function:

$$A_{\rm DW}(E,g) = \frac{1}{3g} + g\left(17E^2+\frac{19}{12}\right) + g^2\left(227E^3+\frac{187E}{4}\right) + g^3\left(\frac{47431}{12}E^4+\frac{34121}{24}E^2+\frac{28829}{576}\right)+\dots$$

- uniform WKB $\rightarrow E = E(B,g)$

Connecting Perturbative/Non-Perturbative Sectors

- perturbative function:

$$E_{\rm DW}(B,g) = B - g\left(3B^2+\frac{1}{4}\right) - g^2\left(17B^3+\frac{19}{4}B\right) - g^3\left(\frac{375}{2}B^4+\frac{459}{4}B^2+\frac{131}{32}\right) - g^4\left(\frac{10689}{4}B^5+\frac{23405}{8}B^3+\frac{22709}{64}B\right) - \dots$$

- non-perturbative function ($\mathcal{F} \sim \exp[-A/2]$):

$$A_{\rm DW}(B,g) = \frac{1}{3g} + g\left(17B^2+\frac{19}{12}\right) + g^2\left(125B^3+\frac{153B}{4}\right) + g^3\left(\frac{17815}{12}B^4+\frac{23405}{24}B^2+\frac{22709}{576}\right) + g^4\left(\frac{87549}{4}B^5+\frac{50715}{2}B^3+\frac{217663}{64}B\right)$$

- simple relation:

$$\frac{\partial E_{\rm DW}}{\partial B} = -6\,B\,g - 3g^2\frac{\partial A_{\rm DW}}{\partial g}$$

Connecting Perturbative/Non-Perturbative Sectors

- similar relations for Sine-Gordon, Fokker-Planck (SUSY DW) and $O(d)$ AHO, ...

- general expression:

$$\frac{\partial E}{\partial B} = -\frac{g}{2S}\left(2B + g\frac{\partial A}{\partial g}\right)$$

- reason: consistency with resurgent trans-series structure at higher non-perturbative order

- implication: non-perturbative function $A(B,g)$ completely determined by perturbative expression $E(B,g)$

$$
\begin{aligned}
f(g^2) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k-1} a_{n,k,l}\, g^{2n}\left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\log\left(-\frac{1}{g^2}\right)\right]^l \\
&= E_{\rm pert}(g^2) + e^{-S/g^2} f_1(g^2) \\
&\quad + e^{-2S/g^2}\left(f_2(g^2) + \ln\left(-\frac{1}{g^2}\right)\tilde{f}_2(g^2)\right) \\
&\quad + e^{-3S/g^2}\left(f_3(g^2) + \ln\left(-\frac{1}{g^2}\right)\tilde{f}_3(g^2) + \ln^2\left(-\frac{1}{g^2}\right)\bar{f}_3(g^2)\right) \\
&\quad + \dots
\end{aligned}
$$

uniform WKB $\Rightarrow$

(i) all f_i come from a single function $\mathcal{F}$

(ii) moreover can be deduced immediately from $E(N, g^2)$

Zinn-Justin/Jentschura: generate entire trans-series from
(i) perturbative expansion $E = E(\nu, g^2)$
(ii) single-instanton fluctuation function $\mathcal{F}(\nu, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

Dunne/Ünsal: perturbation theory generates everything!

$$\mathcal{F}(\nu, g^2) = \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4} \left(\frac{\partial E}{\partial \nu} - 1 + \frac{\left(\nu + \frac{1}{2}\right)\, g^2}{S}\right)\right]$$

dramatic implication: all orders of the multi-instanton trans-series are encoded in perturbation theory of the fluctuations about the perturbative vacuum !!!

why ? turn to path integrals ....

Lecture 4

- Darboux's theorem and resurgent steepest descents analysis
- QM resurgence in terms of saddles
- analytic continuation and complex saddles
- non-perturbative physics without instantons

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard, 1865 - 1963

Analytic Continuation of Path Integrals: Darboux Theorem

- zero dimensions: all-orders steepest descents of contour integrals (Berry/Howls: *hyperasymptotics*)

$$I^{(n)}(k) = \int_{C_n} dz\, e^{-k\, f(z)}$$

- separate out fluctuations:

$$I^{(n)}(k) = \frac{1}{\sqrt{k}}\, e^{-k\, f_n}\, T^{(n)}(k) \quad , \quad T^{(n)}(k) \equiv \sqrt{k} \int_{C_n} dz\, e^{-k\,(f(z)-f_n)}$$

- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r}$$

Analytic Continuation of Path Integrals: Darboux Theorem

- singulant variable: $u \equiv k\,(f(z) - f_n)$

M. V. Berry and C. J. Howls

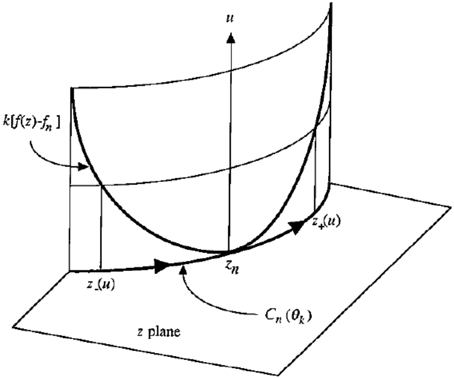
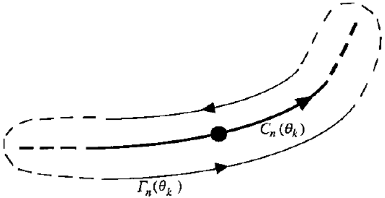


Figure 1. Double-valued mapping (equation (4)) from z to u.



. Steepest path $C_n(\theta_k)$ through saddle n, and loop $\Gamma_n(\theta_k)$ enclos

Analytic Continuation of Path Integrals: Darboux Theorem

- singulant variable:

$$u \equiv k\,(f(z) - f_n)$$

- noting double-valuedness

$$\begin{aligned} T^{(n)}(k) &= \int_0^\infty du\, \frac{e^{-u}}{\sqrt{k}} \left(\frac{1}{f'(z_+(u))} - \frac{1}{f'(z_-(u))} \right) \\ &= \frac{1}{2\pi\, i} \int_0^\infty du\, \frac{e^{-u}}{\sqrt{u}} \oint_{\Gamma_n} dz\, \frac{\sqrt{f(z) - f_n}}{f(z) - f_n - u/k} \end{aligned}$$

- now expand in $\frac{1}{k}$ $\Rightarrow$ fluctuation coefficients:

$$T_r^{(n)} = \frac{\left(r - \frac{1}{2}\right)!}{2\pi\, i} \oint_{\Gamma_n} dz\, \frac{1}{(f(z) - f_n)^{r+1/2}}$$

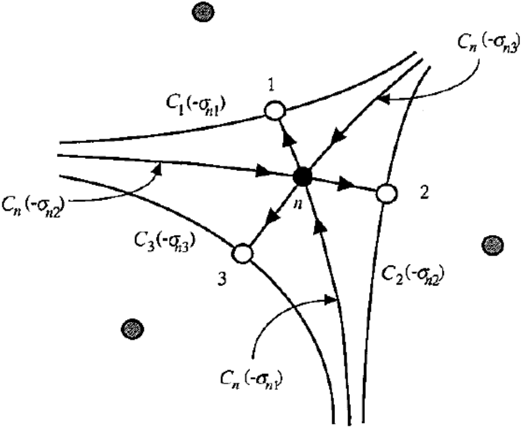
- universal factorial divergence of fluctuations (Darboux)

Analytic Continuation of Path Integrals: Darboux Theorem

- deforming contours:

$$\oint_{\Gamma_n} dz(\ldots) = \sum_{m\,\text{adjacent}} (-1)^{\gamma_{nm}} \int_{C_m} dz\,(\ldots)$$

deforming contours:



Analytic Continuation of Path Integrals: Darboux Theorem

- deforming contours:

$$\oint_{\Gamma_n} dz(\ldots) = \sum_{m\,\text{adjacent}} (-1)^{\gamma_{nm}} \int_{C_m} dz\,(\ldots)$$

- new singulant variables along each contour C_m:

$$T^{(n)}(k) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - v/(k\, F_{nm})} T^{(m)}\left(\frac{v}{F_{nm}}\right)$$

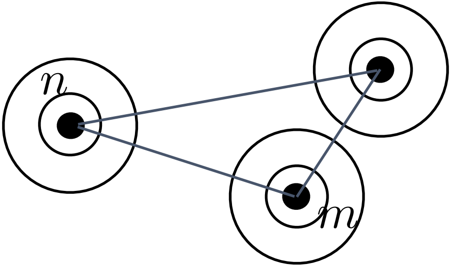
- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

expand fluctuations $T^{(n)}(k) = \sum_r \frac{T_r^{(n)}}{k^r} \quad \Rightarrow$

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \ldots \right.$$

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

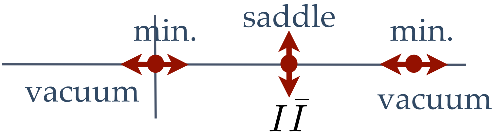
J. Écalle, 1980



zero dim. partition function for periodic potential $V(z) = \sin^2(z)$:

$$I(k) = \int_0^{\pi} dz \, e^{-k \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



Analytic Continuation of Path Integrals: Darboux Theorem

- zero dim. partition function for periodic potential $V(z) = \sin^2(z)$:

$$I(k) = \int_0^\pi dz\, e^{-k\, \sin^2(z)}$$

- two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.

$$I^{(0)}(k) = \frac{1}{\sqrt{k}}\, T^{(0)}(k) \quad , \quad T^{(0)}(k) = \sqrt{k} \int_0^\infty \frac{du}{\sqrt{u}} \frac{e^{-k\,u}}{\sqrt{1-u}} = \sum_{n=0}^\infty \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(n+1)} \frac{1}{k^n}$$

- factorially divergent, as expected, and non-alternating

$$I^{(1)}(k) = \frac{e^{-k}}{\sqrt{k}}\, T^{(1)}(k) \quad , \quad T^{(1)}(k) = i\,\sqrt{k} \int_0^\infty \frac{du}{\sqrt{u}} \frac{e^{-k\,u}}{\sqrt{1+u}} = i \sum_{n=0}^\infty \frac{(-1)^n\,\Gamma\left(n+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(n+1)} \frac{1}{k^n}$$

Analytic Continuation of Path Integrals: Darboux Theorem

- large order behavior about saddle z_0:

$$
\begin{aligned}
T_r^{(0)} &= \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)} \\
&\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1-\frac{1}{4r}+\frac{1}{32r^2}+\frac{1}{128r^3}+\dots\right) \\
&\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1-\frac{1/4}{(r-1)}+\frac{9/32}{(r-1)(r-2)}-\frac{75/128}{(r-1)(r-2)(r-3)}+\dots\right)
\end{aligned}
$$

- low order coefficients about saddle z_1:

$$
T^{(1)}(k) \sim i\sqrt{\pi}\left(1-\frac{1}{4k}+\frac{9}{32k^2}-\frac{75}{128k^3}+\dots\right)
$$

- fluctuations about the two saddles are explicitly related
- resurgence at work!

Resurgence in Path Integrals: "Functional Darboux Theorem"

- periodic potential: $V(x) = \frac{1}{g^2}\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2}g^2 - \frac{13}{8}g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6}g^2 - \frac{1277}{72}g^4 - \dots\right)$$

Resurgence in Path Integrals: “Functional Darboux Theorem”

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle

“functional Darboux theorem”

Resurgence from path integral perspective

- semiclassical expansion of path integral

$$\mathcal{Z}(g^2) = \int \mathcal{D}\phi \, e^{-S[\phi]} \approx \sum_{\text{saddles}\, k} F_k(g^2)\, e^{-\frac{1}{g^2} S_k}$$

Resurgence: asymptotic expansions around different saddles of path integral influence one another

- in principle exact

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$Z = \int dx\, e^{-S(x)}$$

- critical points (saddle points): $\partial S/\partial z = 0$
- steepest descent contour: $\mathrm{Im}\, S(z) = \text{constant}$
- contour flow-time parameter t:

$$\frac{d}{dt}\mathrm{Im}\, S(z) = \frac{1}{2i}\left(\frac{\partial S}{\partial z}\,\dot{z} - \frac{\partial \bar{S}}{\partial \bar{z}}\,\dot{\bar{z}}\right) \quad , \quad \frac{d}{dt}\mathrm{Re}\, S(z) = \frac{1}{2}\left(\frac{\partial S}{\partial z}\,\dot{z} + \frac{\partial \bar{S}}{\partial \bar{z}}\,\dot{\bar{z}}\right)$$

- flow along a steepest decent path:

$$\dot{z} = \frac{\partial \bar{S}}{\partial \bar{z}} \qquad \Rightarrow \frac{d}{dt}\mathrm{Im}\, S(z) = 0 \quad , \quad \frac{d}{dt}\mathrm{Re}\, S(z) = \left|\frac{\partial S}{\partial z}\right|^2 > 0$$

- monotonic in real part

$$Z = e^{-S_{\mathrm{imag}}(x)} \int_\Gamma dz\, e^{-S_{\mathrm{real}}(z)}$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

functional version: path integral

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}\left(S_{\rm real}[A]+i\, S_{\rm imag}[A]\right)} \sim \sum_{\text{thimbles}\, k} e^{-\frac{i}{g^2}\, S_{\rm imag}[A]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2} S_{\rm real}[A]}$$

thimble = functional [configurational] steepest descents contour

remaining path integral has real measure: amenable to
(i) Monte Carlo
(ii) semiclassical expansion (resurgent relations between thimbles)

resurgence: asymptotic expansions about different saddles are closely related

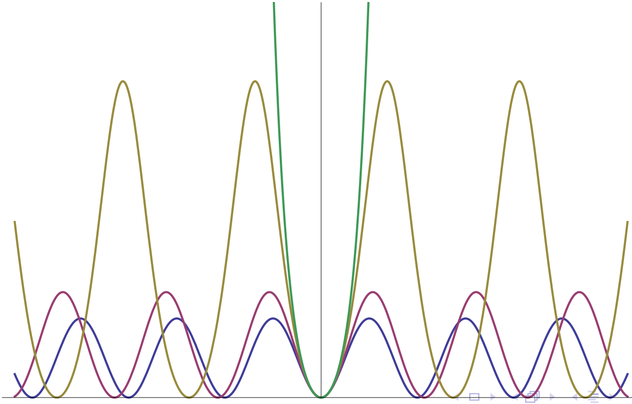
requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Path integrals with complex saddles: “ghost instantons”

- elliptic potential: (Basar, GD, Ünsal, arXiv:1308.1108)

$$V(z|m) = \mathrm{sd}^2(x|m)$$

interpolates between Sine-Gordon ($m = 0$ and Sinh-Gordon ($m = 1$)



Path integrals with complex saddles: zero dim. prototype

$$V(z|m) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,z|m)$$

- duality property:

$$V(z|m)|_{g^2} = V(z|1-m)|_{-g^2}$$

- perturbative series $\sum_n a_n(m) g^{2n}$ satisfies duality:

$$a_n(m) = (-1)^n a_n(1-m)$$

d=0 partition function:

$$\mathcal{Z}(g^2|m) = \frac{1}{g\sqrt{\pi}} \int_{-\mathbb{K}}^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\,\mathrm{sd}^2(z|m)}$$

Path integrals with complex saddles: zero dim. prototype

$$\mathcal{Z}(g^2|0)\Big|_{\rm pert} = 1+\frac{g^2}{4}+\frac{9g^4}{32}+\frac{75g^6}{128}+\frac{3675g^8}{2048}+\frac{59535g^{10}}{8192}+\ldots$$

$$\mathcal{Z}\left(g^2|1\right)\Big|_{\rm pert} = 1-\frac{g^2}{4}+\frac{9g^4}{32}-\frac{75g^6}{128}+\frac{3675g^8}{2048}-\frac{59535g^{10}}{8192}+\ldots$$

$$\mathcal{Z}\left(g^2\middle|\frac{1}{4}\right)\Bigg|_{\rm pert} = 1+\frac{g^2}{8}+\frac{9g^4}{64}+\frac{105g^6}{512}+\frac{1995g^8}{4096}+\frac{48195g^{10}}{32768}+\ldots$$

$$\mathcal{Z}\left(g^2\middle|\frac{3}{4}\right)\Bigg|_{\rm pert} = 1-\frac{g^2}{8}+\frac{9g^4}{64}-\frac{105g^6}{512}+\frac{1995g^8}{4096}-\frac{48195g^{10}}{32768}+\ldots$$

$$\mathcal{Z}\left(g^2\middle|\frac{1}{2}\right)\Bigg|_{\rm pert} = 1+0g^2+\frac{3g^4}{32}+0g^6+\frac{315g^8}{2048}+0g^{10}+\ldots$$

- duality relation: $\mathcal{Z}(g^2|m)=\mathcal{Z}(-g^2|1-m)$

  non-alternating for $m<\frac{1}{2}$ alternating for $m>\frac{1}{2}$

puzzles: Borel summable? "instantons" ?

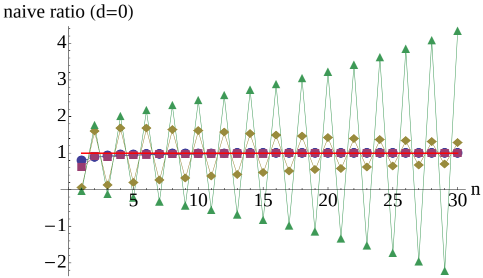
Path integrals with complex saddles: zero dim. prototype

$$\mathcal{Z}(g^2|m) = \frac{2}{g\sqrt{\pi}} \int_0^{\mathbb{K}} dz\, e^{-\frac{1}{g^2}\,\mathrm{sd}^2(z|m)}$$

- large-order behavior about 0 from saddle point $B = \mathbb{K}$:

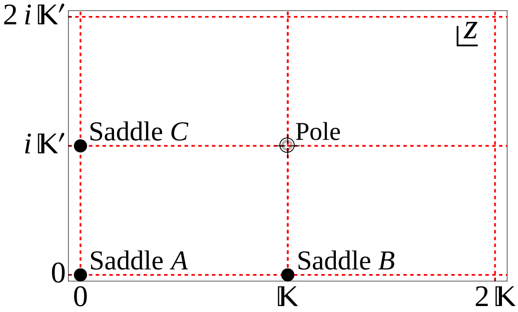
$$S_B = \frac{1}{1-m} \qquad \Rightarrow \qquad a_n \sim \frac{(n-1)!}{\pi S_B^{n+1/2}}$$

- compare with actual series:



disaster !

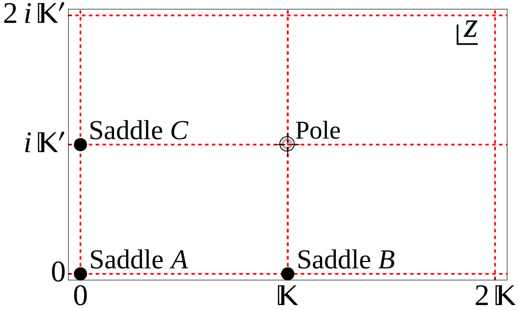
resolution: another saddle off the integration path!



$$S_C = -1/m \qquad \Rightarrow \quad a_n \sim \frac{(n-1)!}{\pi}\left(S_B^{n+1/2} + S_C^{n+1/2}\right)$$

Path integrals with complex saddles: zero dim. prototype

resolution: another saddle off the integration path!

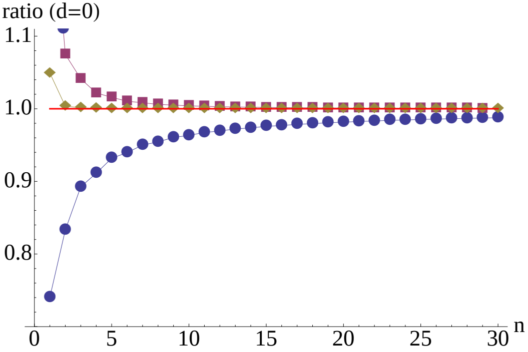


$$S_C = -1/m \qquad \Rightarrow \qquad a_n \sim \frac{(n-1)!}{\pi}\left(S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2}\right)$$

Path integrals with complex saddles: zero dim. prototype

$$a_n \sim \frac{(n-1)!}{\pi}\left(S_B^{n+1/2} + (-1)^n |S_C|^{n+1/2}\right)$$

$\Rightarrow$ improved asymptotics:



conclusion: perturbation series feels *all* saddles, both real and complex

Path integrals with complex saddles: zero dim. prototype

the bigger picture:

• associated with each critical point z_i, there is a unique integration cycle $\mathcal{J}_i$, called a *Lefschetz thimble*, along which the phase remains stationary

• around each saddle there is a contribution of the form:

$$\mathcal{I}^{(k)}(\xi|m) = \frac{1}{\sqrt{\pi}}\sqrt{\xi}\int_{\mathcal{J}_k} dz\, e^{-\xi\,\mathrm{sd}^2(z|m)}$$

• expansions around different saddles are connected via

exact resurgence relation:

$$\mathcal{I}^{(A)}\left(\frac{1}{g^2}|m\right) = \frac{2}{2\pi i}\sum_{k\in\{B,C\}}\int_0^\infty \frac{dv}{v}\frac{1}{1-g^2\,v}\mathcal{I}^{(k)}(v|m)$$

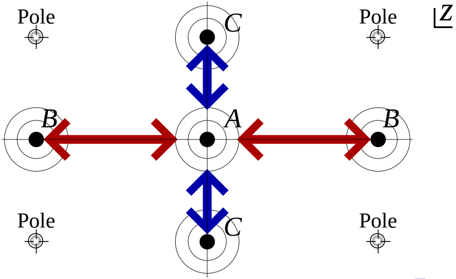
Path integrals with complex saddles: zero dim. prototype

- most general expansion is a three-term trans-series

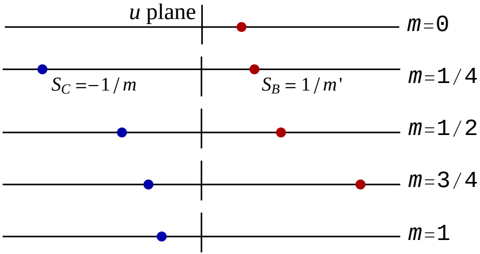
$$\mathcal{Z}_{\mathfrak{C}}(g^2|m) \quad \equiv \quad \sigma_A\, \Phi_A(g^2) + \sigma_B\, e^{-S_B/g^2}\, \Phi_B(g^2) + \sigma_C\, e^{-S_C/g^2}\, \Phi_C(g^2)$$

- coefficients of perturbative expansions are connected

$$a_n^{(A)}(m) = \sum_{j=0} \frac{(n-j-1)!}{\pi} \left(\frac{a_j^{(B)}(m)}{S_B^{\,n-j}} + \frac{a_j^{(C)}(m)}{S_C^{\,n-j}} \right)$$



view from the Borel plane:



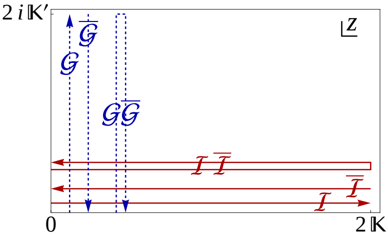
• 'distance' in Borel plane, $\Delta S = S_i - S_j$ ("relative action") controls divergence of perturbation series Φ_j

• $m > 1/2$: closest singularity on $\mathbb{R}^-$ $\Leftrightarrow$ alternating series Φ_A

• mimics structure of both UV and IR renormalons quantum mechanics: ordinary integral $\longrightarrow$ path integral

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}\phi \, e^{-S[\phi]} = \int \mathcal{D}\phi \, e^{-\int d\tau \left(\frac{1}{4}\dot{\phi}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,\phi|m)\right)}$$

- find *real* and *ghost* instantons



- actions:

$$\frac{S_{\mathcal{I}}(m)}{g^2} = \frac{2\sin^{-1}(\sqrt{m})}{g^2\sqrt{mm'}} \geq \frac{2}{g^2} \quad , \quad \frac{S_{\mathcal{G}}(m)}{g^2} = \frac{2\sin^{-1}(\sqrt{m'})}{g^2\sqrt{mm'}} \leq -\frac{2}{g^2}$$

Ghost Instantons: Quantum Mechanical Path Integrals

$$
\begin{aligned}
E^{(0)}(g^2|0) &= 1-\frac{g^2}{4}-\frac{g^4}{16}-\frac{3g^6}{64}-\frac{53g^8}{1024}-\frac{297g^{10}}{4096}-\dots \\
E^{(0)}(g^2|1) &= 1+\frac{g^2}{4}-\frac{g^4}{16}+\frac{3g^6}{64}-\frac{53g^8}{1024}-\frac{297g^{10}}{4096}-\dots \\
E^{(0)}\left(g^2\middle|\frac{1}{4}\right) &= 1-\frac{g^2}{8}-\frac{11g^4}{128}-\frac{3g^6}{128}-\frac{889g^8}{32768}-\frac{225g^{10}}{8192}-\dots \\
E^{(0)}\left(g^2\middle|\frac{3}{4}\right) &= 1+\frac{g^2}{8}-\frac{11g^4}{128}+\frac{3g^6}{128}-\frac{889g^8}{32768}+\frac{225g^{10}}{8192}-\dots \\
E^{(0)}\left(g^2\middle|\frac{1}{2}\right) &= 1+0g^2-\frac{3g^4}{32}+0g^6-\frac{39g^8}{2048}+0g^{10}-\dots
\end{aligned}
$$

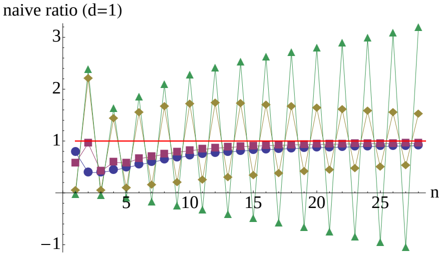
- duality relation: $E^{(0)}(g^2|m)=E^{(0)}(-g^2|1-m)$

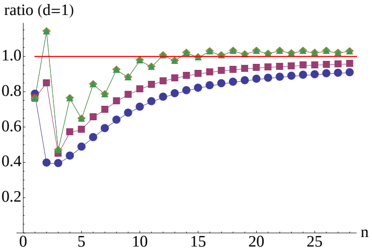
  non-alternating for $m<\frac{1}{2}$ alternating for $m>\frac{1}{2}$

- very similar to zero-dimensional protoype!

Ghost Instantons: Quantum Mechanical Path Integrals

- large order growth of QM perturbation theory





without ghost instantons *with ghost instantons*

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

the bigger picture:

- vacuum "talks to" the topologically trivial sector:

$$\ldots \leftrightarrow [\mathcal{G}^2\bar{\mathcal{G}}^2] \quad \leftrightarrow \quad [\mathcal{G}\bar{\mathcal{G}}] \quad \leftrightarrow \quad \text{pert.vac} \quad \leftrightarrow \quad [\mathcal{I}\bar{\mathcal{I}}] \quad \leftrightarrow [\mathcal{I}^2\bar{\mathcal{I}}^2] \leftrightarrow \ldots$$

- QM trans-series:

$$\mathcal{Z}(g^2|m) = \begin{cases} \Phi_0(g^2) + [\mathcal{I}\bar{\mathcal{I}}]_- \Phi_{[\mathcal{I}\bar{\mathcal{I}}]}(g^2) + [\mathcal{I}^2\bar{\mathcal{I}}^2]_- \Phi_{[\mathcal{I}^2\bar{\mathcal{I}}^2]}(g^2) + \ldots & -\pi < arg(g^2) < 0 \\ \Phi_0(g^2) + [\mathcal{I}\bar{\mathcal{I}}]_+ \Phi_{[\mathcal{I}\bar{\mathcal{I}}]}(g^2) + [\mathcal{I}^2\bar{\mathcal{I}}^2]_+ \Phi_{[\mathcal{I}^2\bar{\mathcal{I}}^2]}(g^2) + \ldots & 0 < arg(g^2) < \pi \end{cases}$$

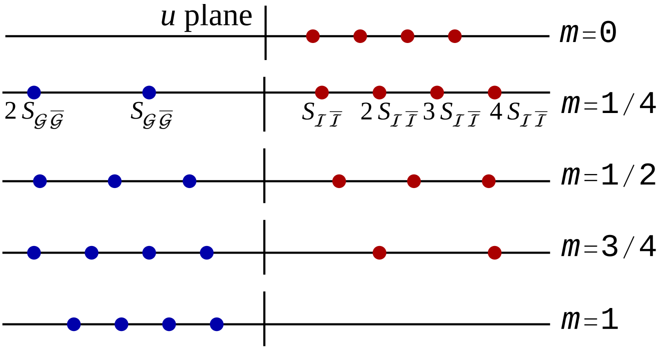
- ambiguities cancel ad-infinitum (resurgence!)

$$\text{Im}\left(\mathcal{S}_{0\pm}\Phi_0 + [\mathcal{I}\bar{\mathcal{I}}]_{0\pm}\text{Re}\mathcal{S}_0\Phi_{[\mathcal{I}\bar{\mathcal{I}}]}\right) = 0 \quad \text{up to} \quad \mathcal{O}(e^{-4S_I})$$

- Similar structure for one instanton, etc.. sector

$$\ldots \leftrightarrow [\mathcal{I}\mathcal{G}^2\bar{\mathcal{G}}^2] \quad \leftrightarrow \quad [\mathcal{I}\mathcal{G}\bar{\mathcal{G}}] \quad \leftrightarrow \quad [\mathcal{I}] \quad \leftrightarrow \quad [\mathcal{I}^2\bar{\mathcal{I}}] \quad \leftrightarrow [\mathcal{I}^3\bar{\mathcal{I}}^2] \leftrightarrow \ldots$$

Borel plane structure:



Mimics IR and UV renormalon structure of asymptotically free QFT

Non-perturbative Physics Without Instantons

e.g, 2d Principal Chiral Model:

(Cherman, Dorigoni, GD, Ünsal, 1308.0127)

$$S_b = \frac{N}{2\lambda} \int d^2x \, \mathrm{tr}\, \partial_\mu U \partial^\mu U^\dagger, \quad U \in SU(N),$$

• non-Borel-summable perturbation theory due to IR renomalons

• but, the theory has no instantons !

resolution: there exist non-BPS saddle point solutions to the second-order classical Euclidean equations of motion: "unitons"

$$\partial_\mu \left(U^\dagger \partial_\mu U \right) = 0$$

Non-perturbative Physics Without Instantons: Principal Chiral Model

- "unitons": $U \equiv (\mathbf{1} - 2\,\mathbb{P})$

$$\partial_\mu \left(U^\dagger \partial_\mu U \right) = 0 \qquad \rightarrow \qquad [\mathbb{P}, \partial_\mu^2 \mathbb{P}] = 0$$

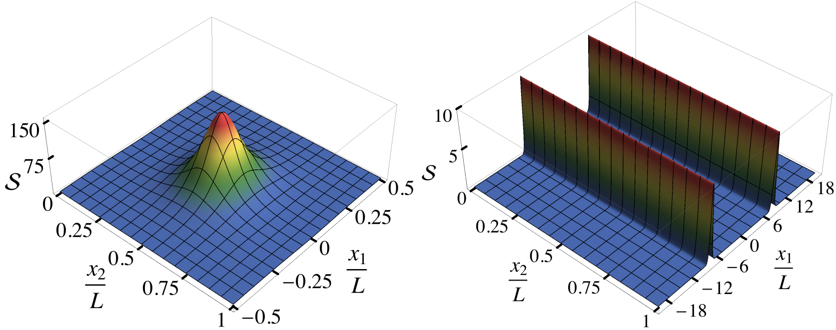
- general solutions to $\mathbb{CP}^{N-1}$ model (Din/Zakrzewski)
- simplest untions: from $\mathbb{CP}^{N-1}$ instantons
- "fractons": twisted & fractionalized solutions in PCM

N fundamental fractons, $U(z,\bar{z}) = e^{i\pi/N}(1-2\mathbb{P})$,
$\mathbb{P}_{ij} = \frac{v_i v_j^\dagger}{v^\dagger \cdot v}$

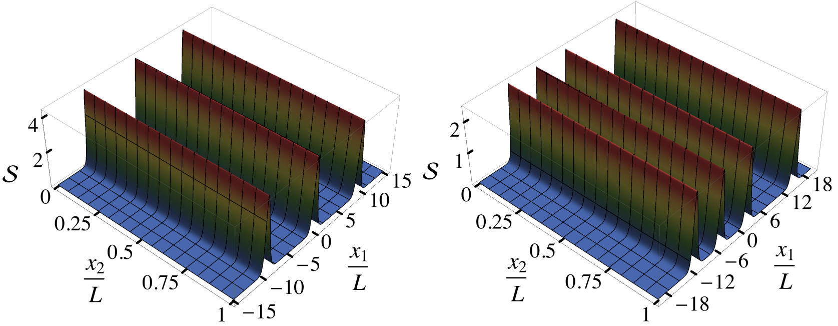
$$S_F = \frac{8\pi}{g^2\,N}$$

Non-perturbative Physics Without Instantons: Principal Chiral Model

$SU(2)$ fractons



$SU(3)$ and $SU(4)$ fractons



Non-perturbative Physics Without Instantons: Principal Chiral Model

$$\mathcal{F}_i \sim e^{-\frac{8\pi(\mu_{i+1}-\mu_i)}{g^2}} \sim e^{-\frac{8\pi}{g^2 N}}, \qquad \mathcal{U} = \prod_{i=1}^{N} \mathcal{F}_i$$

- perturbation theory: IR renormalon singularities on positive Borel axis

$$t_k^+ = 8\pi k/N = k[g^2 S_{\mathcal{U}}]/\beta_0, \qquad k \in \mathbb{Z}^+$$

- ambiguous lateral Borel sum:

$$\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \Re\,\mathbb{B}_0 \mp i\,\frac{32\pi}{g^2 N} e^{-\frac{16\pi}{g^2 N}}$$

- non-perturbative fracton/anti-fracton amplitude:

$$[\mathbb{F}_i \bar{\mathbb{F}}_i]_{\theta=0^\pm} = \left[\log\left(\frac{g^2 N}{16\pi}\right) - \gamma\right] \frac{16}{g^2 N} e^{-\frac{16\pi}{g^2 N}} \pm i\,\frac{32\pi}{g^2 N} e^{-\frac{16\pi}{g^2 N}}$$

Non-perturbative Physics Without Instantons

Yang-Mills, $\mathbb{CP}^{N-1}$, PCM, ... all have non-BPS solutions with finite action

- "unstable": negative modes of fluctuation operator
- what do these mean ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Non-perturbative Physics Without Instantons: Yang-Mills

4d Yang-Mills: $S_{YM} = \frac{1}{2}\int d^4x \,\mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu}\right)$

- Bogomolny factorization:

$$S_{YM} = \frac{1}{4}\int d^4x \,\mathrm{tr}\left\{\left(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\right)^2 \pm 2F_{\mu\nu}\tilde{F}_{\mu\nu}\right\}$$

- classical equations of motion:

$$D_\mu F_{\mu\nu} = 0 \quad \longrightarrow \quad F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$$

- "instantons": minima of classical action

- non-BPS finite action saddle-points: Sibner, Sibner, Uhlenbeck ($SU(2)$): locally m instantons & m anti-instantons, $m \in \mathbb{Z} \geq 2$

- ansatz constructions for $SU(n)$, $n \geq 3$

- solutions 'unstable' : negative modes

(Dabrowski, GD, arXiv:1306.0921)

Non-self-dual Solutions in $\mathbb{CP}^{N-1}$

$$S = \int d^2x \left[\frac{1}{2} \left| D_\mu n \pm i\epsilon_{\mu\nu} D_\nu n \right|^2 \mp i\epsilon_{\mu\nu} \left(D_\nu n\right)^\dagger D_\mu n \right]$$

- rank-1 projector representation:

$$\mathbb{P} \equiv n\, n^\dagger$$

$\mathbb{P}^2 = \mathbb{P} = \mathbb{P}^\dagger$, $\mathrm{Tr}\, \mathbb{P} = 1$

$$\begin{aligned} \text{action} \quad S &= 2 \int d^2x \, \mathrm{Tr} \left[\partial_z \mathbb{P} \, \partial_{\bar{z}} \mathbb{P} \right] \\ \text{charge} \quad Q &= 2 \int d^2x \, \mathrm{Tr} \left[\mathbb{P} \, \partial_{\bar{z}} \mathbb{P} \, \partial_z \mathbb{P} - \mathbb{P} \, \partial_z \mathbb{P} \, \partial_{\bar{z}} \mathbb{P} \right] \end{aligned}$$

Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

- first-order instanton equations:

$$D_\mu n = \pm\, i\epsilon_{\mu\nu} D_\nu n$$

$$\partial_{\bar z}\mathbb{P}\,\mathbb{P} = 0 \quad \text{(instanton)} \quad , \quad \partial_z\mathbb{P}\,\mathbb{P} = 0 \quad \text{(anti-instanton)}$$

- solution: holomorphic projector $\mathbb{P} = \frac{\omega\,\omega^\dagger}{\omega^\dagger\omega}$, with $\omega = \omega(z)$

second-order classical equations:

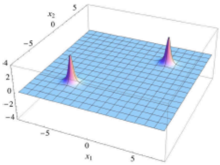
$$D_\mu D_\mu n - (n^\dagger \cdot D_\mu D_\mu n)\, n = 0 \qquad \text{or} \qquad [\partial_z \partial_{\bar z}\mathbb{P}\,,\, \mathbb{P}] = 0$$

- non-BPS solutions generated from instantons:

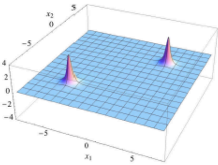
$$Z_+ : \omega \to Z_+\omega \equiv \partial_z\,\omega - \frac{(\omega^\dagger\,\partial_z\,\omega)}{\omega^\dagger\omega}\,\omega \quad , \quad Z_+ : n \to Z_+ n \equiv \frac{Z_+\omega}{|Z_+\omega|}$$

$$\omega_{(0)} \xrightarrow{Z_+} \omega_{(1)} \xrightarrow{Z_+} \cdots \xrightarrow{Z_+} \omega_{(k)} \xrightarrow{Z_+} \cdots\ \omega_{(N-1)} \xrightarrow{Z_+} 0$$

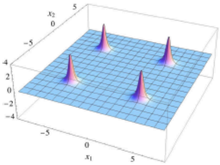
Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$



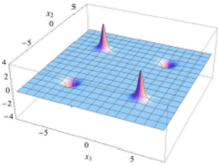
Action Density of $\omega_{(0)}$: $(S_{(0)} = 2)$



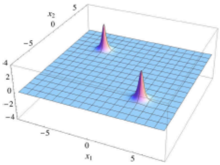
Charge Density of $\omega_{(0)}$: $(Q_{(0)} = 2)$



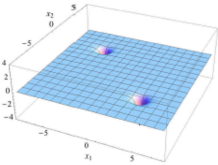
Action Density of $\omega_{(1)}$: $(S_{(1)} = 4)$



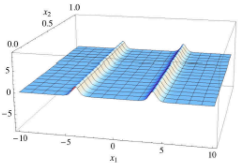
Charge Density of $\omega_{(1)}$: $(Q_{(1)} = 0)$



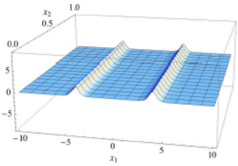
Action Density of $\omega_{(1)}$: $(S_{(2)} = 2)$



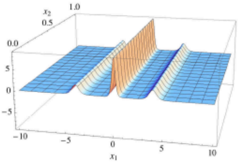
Charge Density of $\omega_{(2)}$: $(Q_{(2)} = -2)$



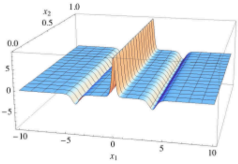
Action Density of $\omega_{(0)}$: $(S_{(0)} = \frac{2}{3})$



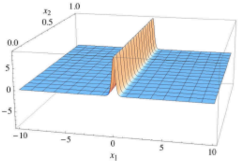
Charge Density of $\omega_{(0)}$: $(Q_{(0)} = \frac{2}{3})$



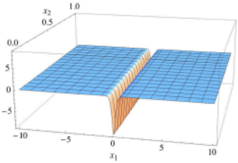
Action Density of $\omega_{(1)}$: $(S_{(1)} = \frac{4}{3})$



Charge Density of $\omega_{(1)}$: $(Q_{(1)} = 0)$



Action Density of $\omega_{(2)}$: $(S_{(2)} = \frac{2}{3})$



Charge Density of $\omega_{(2)}$: $(Q_{(2)} = -\frac{2}{3})$

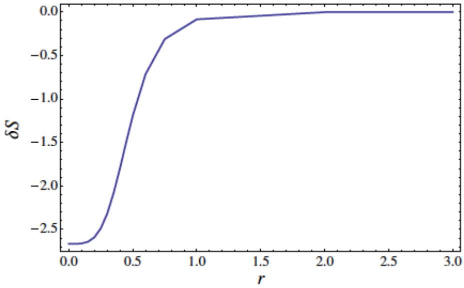
Non-perturbative Physics Without Instantons: $\mathbb{CP}^{N-1}$

- non-BPS solutions are 'unstable': e.g.

$$n \to \tilde{n} = n\sqrt{1-\phi^\dagger\phi} + \phi \quad , \quad \phi = D_z n \quad ; \quad \phi^\dagger \cdot n = 0$$

- change in action is manifestly negative:

$$\delta S = -\int d^2x \left(\mathrm{Tr}\left[(D_z n)^\dagger D_z n \, (D_{\bar z} n)^\dagger D_{\bar z} n \right] + \mathrm{Tr}\left[(D_{\bar z} n)^\dagger D_z n \, (D_z n)^\dagger D_{\bar z} n \right] \right)$$



physical origin of negative modes:

• single $\mathbb{CP}^{N-1}$ instanton: $2N$ parameters: i.e. $2N$ zero modes

• $Q = 2$ $\mathbb{CP}^{N-1}$ instanton: $4N$ parameters: i.e. $4N$ zero modes

• mapped non-BPS solution also has $4N$ parameters: i.e. $4N$ zero modes

• but, "looks like" 2 instantons and 2 anti-instantons $\Rightarrow$ $8N$ zero modes

$\Rightarrow$ $4N$ zero modes are lifted at finite separation

some become negative modes

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative world
- there is extra ‘magic’ in perturbation theory
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- basic property of steepest descents expansions
- basic property of complex differential equations
- trans-series: sectors are inter-related
- resurgence triangle: network of connections
- moral: consider all saddles, including non-BPS
- resurgence required for analytic continuation

Open Problems

- Resurgence in Chern-Simons theories, Euler-Heisenberg, dS/AdS, exact S-matrices, matrix models, topological strings, integrability, localization, ...
- nonlinear differential equations
- natural path integral construction
- analytic continuation of path integrals
- ODE/IM correspondence
- relating strong- and weak-coupling expansions: dualities
- relation to SUSY and extended SUSY
- ...

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