Resurgence and Trans-series in Quantum Theories

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GD & Mithat Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

GD, lectures at CERN 2014 Winter School

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108, 1405.0302, 1501.05671

Lecture 1

- motivation: physical and mathematical
- definition of resurgent trans-series
- divergence of perturbation theory in QM
- basics of Borel summation
- the Bogomolny/Zinn-Justin cancellation mechanism

Physical Motivation

• infrared renormalon puzzle in asymptotically free QFT

• non-perturbative physics without instantons: physical meaning of non-BPS saddles

<u>Bigger Picture</u>

- strongly interacting/correlated systems
- non-perturbative definition of non-trivial QFT in continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- uncover hidden 'magic' in perturbation theory
- "exact" asymptotics in QM, QFT and string theory

Physical Motivation

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$

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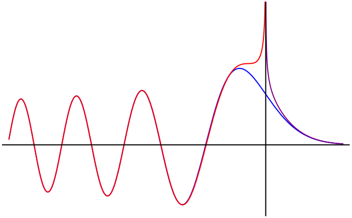


$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}t^3 + x\,t\right)}\, dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\[2ex] \dfrac{\sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view semiclassical expansions as potentially exact

Resurgent Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

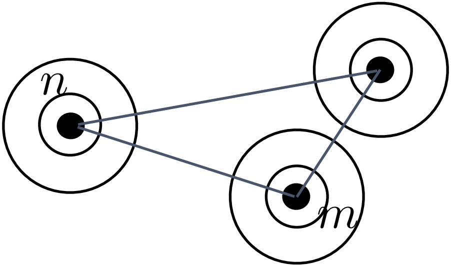
- J. Écalle (1980): set of functions closed under:

(Borel transform) + (analytic continuation) + (Laplace transform)

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponentially improved asymptotics

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Perturbation theory

- hard problem = easy problem + "small" correction
- perturbation theory generally → divergent series

e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

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e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients but it works ...

Perturbation theory works

QED perturbation theory:

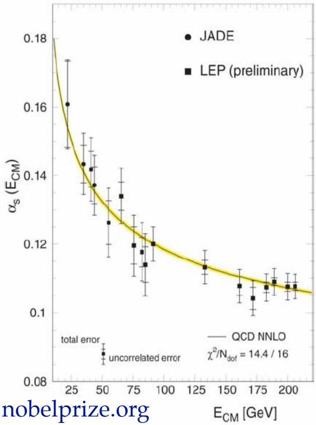
$$\frac{1}{2}(g-2) = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - (0.32848\ldots)\left(\frac{\alpha}{\pi}\right)^2 + (1.18124\ldots)\left(\frac{\alpha}{\pi}\right)^3 - (1.7283(35))\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{exper}} = 0.001\,159\,652\,180\,73(28)$$

$$\left[\tfrac{1}{2}(g-2)\right]_{\text{theory}} = 0.001\,159\,652\,184\,42$$

QCD: asymptotic freedom

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N_C - \frac{4}{3}\frac{N_F}{2}\right)$$



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Perturbation theory

but it is divergent ...

Perturbation theory: divergent series

Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever ... That most of these things [summation of divergent series] are correct, in spite of that, is extraordinarily surprising. I am trying to find a reason for this; it is an exceedingly interesting question.

N. Abel, 1802 – 1829

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The series is divergent; therefore we may be able to do something with it

O. Heaviside, 1850 – 1925

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n \, (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

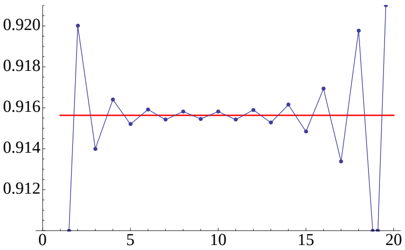
$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

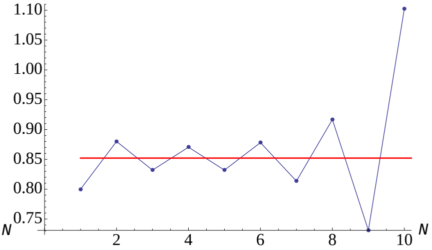
truncate just before least term (x dependent!)

Asymptotic Series vs Convergent Series

$$\sum_{n=1}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$



$(x = 0.1)$

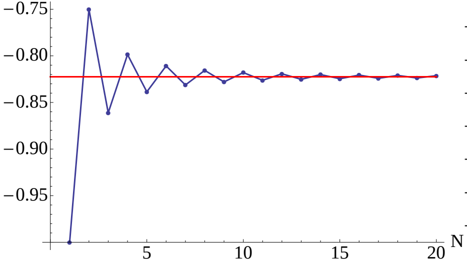


$(x = 0.2)$

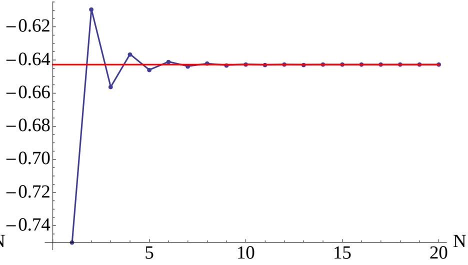
optimal truncation order depends on x: $N_{\text{opt}} \approx \frac{1}{x}$

contrast with behavior of a convergent series:
more terms always improves the answer, independent of x

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}\, x^n = \mathrm{PolyLog}(2, -x)$$



$(x = 1)$



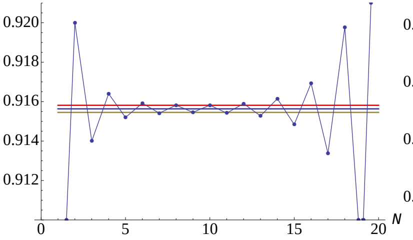
$(x = 0.75)$

Asymptotic Series: exponential precision

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$

optimal truncation: error term is <u>exponentially small</u>

$$|R_N(x)|_{N\approx 1/x} \approx N!\, x^N\Big|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$



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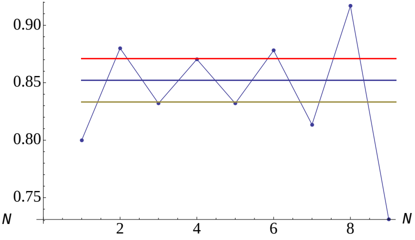
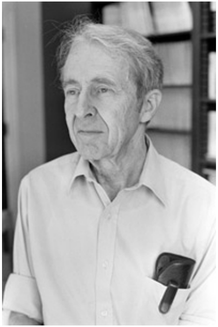


$(x = 0.2)$

Asymptotic Series vs Convergent Series

Divergent series converge faster than convergent series because they don't have to converge

G. F. Carrier, 1918 – 2002



Borel summation: basic idea

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

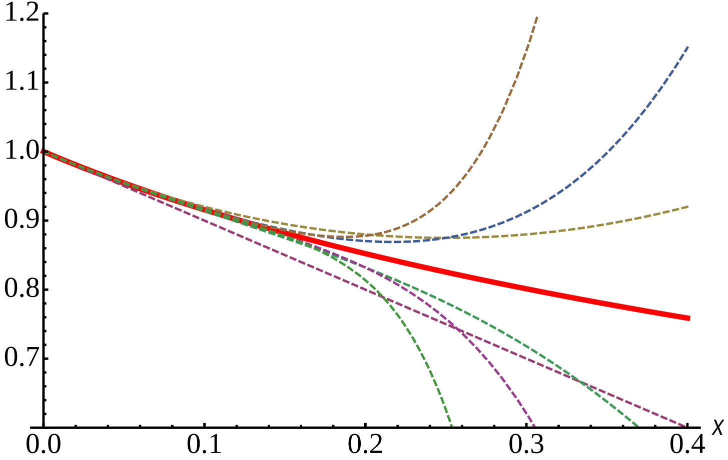
alternating factorially divergent series:


Emile Borel

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, g^n = \int_0^\infty dt\, e^{-t} \frac{1}{1+g\,t} \qquad (?)$$

integral convergent for all $g > 0$: "Borel sum" of the series

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+x\,t}$$



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non-alternating factorially divergent series:

$$\sum_{n=0}^{\infty} n!\, g^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1 - g\, t} \qquad (??)$$

pole on the Borel axis!

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pole on the Borel axis!

Emile Borel

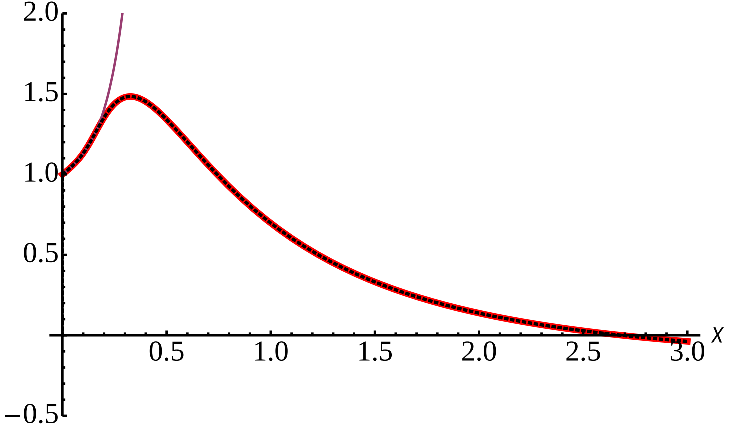
$\Rightarrow$ non-perturbative imaginary part

$$\pm \frac{i\,\pi}{g}\, e^{-\frac{1}{g}}$$

but every term in the series is real !?!

Borel Summation: basic idea

$$\text{Borel} \quad \Rightarrow \quad \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^{\infty} dt\, e^{-t}\, \frac{1}{1-x\,t} = \frac{1}{x}\, e^{-\frac{1}{x}}\, Ei\left(\frac{1}{x}\right)$$



Borel summation

Borel transform of series $f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n$:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has finite radius of convergence.

Borel resummation of original asymptotic series:

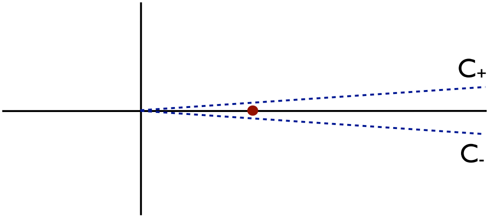
$$\mathcal{S} f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

warning: $\mathcal{B}[f](t)$ may have singularities in (Borel) t plane

Borel singularities

avoid singularities on $\mathbb{R}^+$: lateral Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t) e^{-t/g} dt$$



go above/below the singularity: $\theta = 0^\pm$

$\longrightarrow$ non-perturbative ambiguity: $\pm \mathrm{Im}[\mathcal{S}_0 f(g)]$

challenge: use physical input to resolve ambiguity

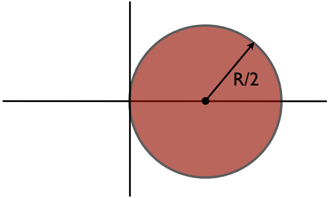
Borel summation: existence theorem (Nevanlinna & Sokal)

$f(z)$ analytic in circle $C_R = \{z : \left|z - \frac{R}{2}\right| < \frac{R}{2}\}$

$$f(z) = \sum_{n=0}^{N-1} a_n\, z^n + R_N(z) \qquad , \qquad |R_N(z)| \leq A\, \sigma^N\, N!\, |z|^N$$

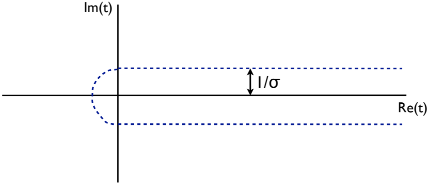
Borel transform

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\, t^n$$



analytic continuation to
$S_\sigma = \{t : |t - \mathbb{R}^+| < 1/\sigma\}$

$$f(z) = \frac{1}{z} \int_0^\infty e^{-t/z}\, B(t)\, dt$$



another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

$\Rightarrow$ the trans-series really IS the function

Resurgence: Preserving Analytic Continuation

- zero-dimensional partition functions

$$Z_1(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2\lambda}\sinh^2(\sqrt{\lambda}\,x)} = \frac{1}{\sqrt{\lambda}}\, e^{\frac{1}{4\lambda}}\, K_0\left(\frac{1}{4\lambda}\right)$$

$$\sim \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(-1)^n(2\lambda)^n\frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{Borel-summable}$$

$$Z_2(\lambda) = \int_{0}^{\pi/\sqrt{\lambda}} dx\, e^{-\frac{1}{2\lambda}\sin^2(\sqrt{\lambda}\,x)} = \frac{\pi}{\sqrt{\lambda}}\, e^{-\frac{1}{4\lambda}}\, I_0\left(\frac{1}{4\lambda}\right)$$

$$\sim \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(2\lambda)^n\frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{non-Borel-summable}$$

- naively: $Z_1(-\lambda) = Z_2(\lambda)$
- connection formula: $K_0(e^{\pm i\pi}\,|z|) = K_0(|z|) \mp i\,\pi\, I_0(|z|)$

Resurgence: Preserving Analytic Continuation

- Borel summation

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\; {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)$$

- lateral Borel summation

$$\begin{aligned}
&Z_1(e^{i\pi}\,\lambda) - Z_1(e^{-i\pi}\,\lambda)\\
&= \sqrt{\frac{\pi}{2}}\frac{1}{2\lambda}\int_1^\infty dt\, e^{-\frac{t}{2\lambda}}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t-i\varepsilon\right) - \;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t+i\varepsilon\right)\right]\\
&\quad = \quad -(2i)\sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\,e^{-\frac{1}{2\lambda}}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)\\
&\quad = \quad -2\,i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)
\end{aligned}$$

- connection formula: $Z_1(e^{\pm i\pi}\,\lambda) = Z_2(\lambda) \mp i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)$

Resurgence: Preserving Analytic Continuation

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots + \frac{174611}{6600z^{20}} - \dots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$

formal series $\Rightarrow$ $\operatorname{Im} \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$

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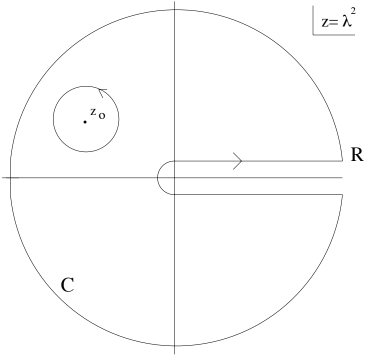
- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi\cot(\pi z)$

$$\Rightarrow \quad \mathrm{Im}\,\psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi\sum_{k=1}^{\infty} e^{-2\pi k y}$$

"raw" asymptotics inconsistent with analytic continuation cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964

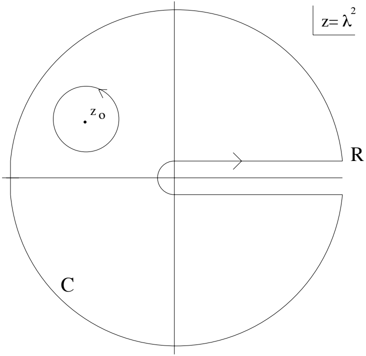


$$E(z_0) = \frac{1}{2\pi i} \oint_C dz\, \frac{E(z)}{z - z_0}$$

$$= \frac{1}{\pi} \int_0^R dz\, \frac{Im\, E(z)}{z - z_0}$$

$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi} \int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}} \right)$$

cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964



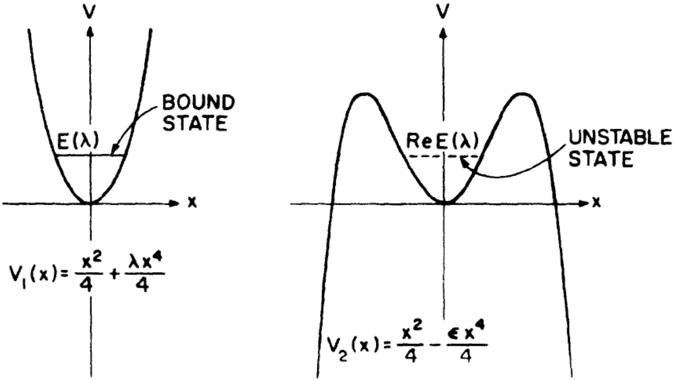
$$E(z_0) = \frac{1}{2\pi i}\oint_C dz\, \frac{E(z)}{z-z_0}$$
$$= \frac{1}{\pi}\int_0^R dz\, \frac{Im\, E(z)}{z-z_0}$$
$$= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}}\right)$$

WKB $\Rightarrow \mathrm{Im}\, E(z) \sim \frac{a}{\sqrt{z}}\, e^{-b/z} \quad , \quad z \to 0$

$$\Rightarrow \qquad c_n \sim \frac{a}{\pi}\int_0^\infty dz\, \frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}} \qquad \checkmark$$

Instability and Divergence of Perturbation Theory

quartic AHO: $V(x) = \frac{x^2}{4} + \lambda \frac{x^4}{4}$ Bender/Wu, 1969



Divergence of perturbation theory

an important part of the story ...

The majority of nontrivial theories are seemingly unstable at some phase of the coupling constant, which leads to the asymptotic nature of the perturbative series

A. Vainshtein (1964)

Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n \, g^n \qquad , \qquad c_n \sim \beta^n \, \Gamma(\gamma \, n + \delta)$$

- alternating series: real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1+t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

- nonalternating series: ambiguous imaginary part

$$\mathrm{Re}\, f(-g) \sim \frac{1}{\gamma} \mathcal{P} \int_0^\infty \frac{dt}{t} \left(\frac{1}{1-t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

$$\mathrm{Im}\, f(-g) \sim \pm \frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
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- periodic Sine-Gordon potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

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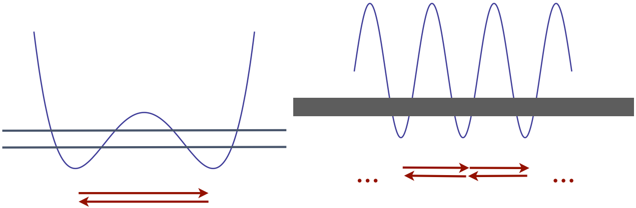
- Stark: $c_n \sim (2n)!$ — unstable
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$ — stable
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$ — unstable
- periodic Sine-Gordon potential: $c_n \sim n!$ — stable ???
- double-well: $c_n \sim n!$ — stable ???

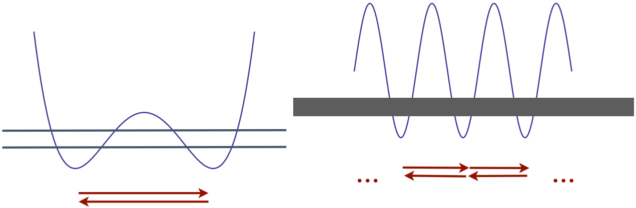
Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

Bogomolny/Zinn-Justin mechanism in QM



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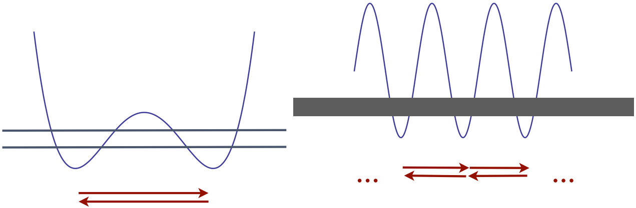
splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel summable:
     ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

Bogomolny/Zinn-Justin mechanism in QM

e.g., double-well: $V(x) = x^2(1 - g\,x)^2$

$$E_0 \sim \sum_n c_n\, g^{2n}$$

- perturbation theory:

$$c_n \sim -3^n\, n! \quad : \quad \text{Borel} \quad \Rightarrow \quad \text{Im}\, E_0 \sim \mp \pi\, e^{-\frac{1}{3g^2}}$$

- non-perturbative analysis: instanton: $g\,x_0(t) = \frac{1}{1+e^{-t}}$
- classical Eucidean action: $S_0 = \frac{1}{6g^2}$
- non-perturbative instanton gas:

$$\text{Im}\, E_0 \sim \pm \pi\, e^{-2\frac{1}{6g^2}}$$

- BZJ cancellation $\Rightarrow$ E_0 is real and unambiguous

"resurgence" $\Rightarrow$ cancellation to all orders

Bogomolny/Zinn-Justin mechanism in QM

- double-well potential: $V(x) = \frac{1}{2}\, x^2\, (1 - g\, x)^2$

approximate $\mathcal{I}\bar{\mathcal{I}}$ soln. : $x_{cl}(t) = \begin{cases} x_0(R+t) & , \quad t > 0 \\ x_0(R-t) & , \quad t < 0 \end{cases}$

effective interaction potential: $U_{\rm int}(t_1, t_2) = -\frac{2}{g^2}\, e^{-|t_1 - t_2|}$

$$
\begin{aligned}
Z_{\rm int} \quad &= \quad a^2 \int dt_1 \int dt_2\, e^{-U_{int}(t_1,t_2)} \qquad \left(a \equiv \frac{1}{g\,\sqrt{\pi}}\, e^{-\frac{1}{6g^2}} \right) \\
&\overset{T\to\infty}{\sim} \quad \frac{1}{2} T^2\, a^2 + T\, a^2 \int_0^\infty dt \left(\exp\left[\frac{2}{g^2} e^{-t}\right] - 1 \right) + \dots
\end{aligned}
$$

- instability: as $g^2 \to 0$, dominated by $t \to 0$???

Bogomolny/Zinn-Justin mechanism in QM

$$Z_{\rm int} \overset{T\to\infty}{\sim} \frac{1}{2}T^2\, a^2 + T\, a^2 \int_0^\infty dt \left(\exp\left[\frac{2}{g^2}e^{-t}\right] - 1\right) + \dots$$

BZJ idea: analytically continue $g^2 \to -g^2$

$\Rightarrow$ dominated by finite t $\Rightarrow$ stable instanton gas

$$\int_0^\infty dt \left(\exp\left[-\frac{2}{g^2}e^{-t}\right] - 1\right) \sim -\gamma_E + \ln\left(\frac{g^2}{2}\right) + Ei\left(-\frac{2}{g^2}\right)$$

- ambiguous imaginary part (from log) when $-g^2 \to g^2$
- recall $Z \sim e^{-E_0\, T}$ $\Rightarrow$ imaginary E_0 from instanton gas

BZJ cancellation: cancels against ambiguous imaginary part from analytic continuation of Borel summation of perturbation theory

Decoding of Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$

- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0}\, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i\, \pi\, \alpha\, e^{-2S/g^2}$

- but $c_{0,2,1} = -\alpha$: BZJ cancellation !

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- but $c_{0,2,1} = -\alpha$: BZJ cancellation !

pert flucs about instanton: $e^{-S/g^2}\, \left(1 + a_1 g^2 + a_2 g^4 + \dots\right)$

divergent:

$a_n \sim \frac{n!}{(2S)^n}\,(a\, \ln n + b) \Rightarrow \pm i\, \pi\, e^{-3S/g^2}\left(a \ln \frac{1}{g^2} + b\right)$

- 3-instanton: $e^{-3S/g^2}\left[\frac{a}{2}\left(\ln\left(-\frac{1}{g^2}\right)\right)^2 + b\, \ln\left(-\frac{1}{g^2}\right) + c\right]$

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resurgence: *ad infinitum*, also sub-leading large-order terms

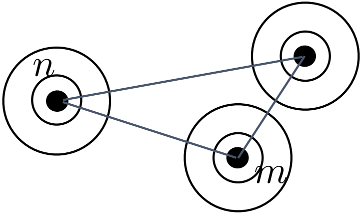
Lecture 2

- divergence of perturbation theory in QFT
- Euler-Heisenberg effective actions & Schwinger effect
- complex instantons and quantum interference
- IR renormalon puzzle in asymptotically free QFT

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$

- perturbation theory is generically asymptotic



- resurgent trans-series

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p} \, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

Towards Resurgence in QFT

- resurgence $\equiv$ analytic continuation of trans-series

- effective actions, partition functions, ..., have natural integral representations with resurgent asymptotic expansions

- analytic continuation of external parameters: temperature, chemical potential, external fields, ...

- e.g., magnetic $\leftrightarrow$ electric; de Sitter $\leftrightarrow$ anti de Sitter, ...

- matrix models, large N, strings, ... (Mariño, Schiappa, ...)

- soluble QFT: Chern-Simons, ABJM, $\rightarrow$ matrix integrals

- asymptotically free QFT ? ... "renormalons"

Divergence from combinatorics

- typical leading growth: $c_n \sim (\pm 1)^n \, \beta^n \, \Gamma(\gamma \, n + \delta)$
- factorial growth of number of Feynman diagrams

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2 - g\,x^4} \, dx = \sum_{n=0}^{\infty} J_n \, g^n \Rightarrow \quad J_n \sim (-1)^n \, (n-1)!$$

- ϕ^4 and ϕ^3: $\quad J_n \sim c^n \, n!$ (Hurst, 1952; Thirring, 1953)
- QED: (Riddell, 1953)

$$J(n, \epsilon, \rho) = \frac{(n!)^2}{(\epsilon!)^2 (n-\epsilon)!} \cdot \frac{n!}{\rho! \, \left[\frac{1}{2}(n-\rho)\right]! \, 2^{(n-\rho)/2}}$$

$\epsilon = \#$ ext. electron lines , $\rho = \#$ ext. photon lines

- comment: large N limit in YM/QCD:

number of planar diagrams grows as a power law!

$$J_n^{\text{planar}} \sim c^n$$

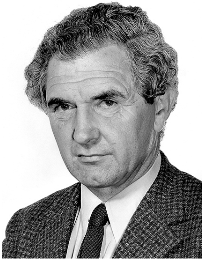
(Koplik, Neveu, Nussinov, 1977)

Divergence of perturbation theory in QFT

C. A. Hurst (1952);
W. Thirring (1953)

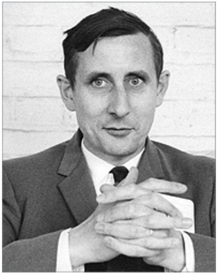
ϕ^4 pert. theory divergent

(i) factorial growth # diagrams
(ii) explicit lower bounds on diagrams



If it be granted that the perturbation expansion does not lead to a convergent series in the coupling constant for all theories which can be renormalized, at least, then a reconciliation is needed between this and the excellent agreement found in electrodynamics between experimental results and low-order calculations. It is suggested that this agreement is due to the fact that the S-matrix expansion is to be interpreted as an asymptotic expansion in the fine-structure constant ...

Dyson's argument (QED)



- F. J. Dyson (1952):
physical argument for divergence of QED perturbation theory

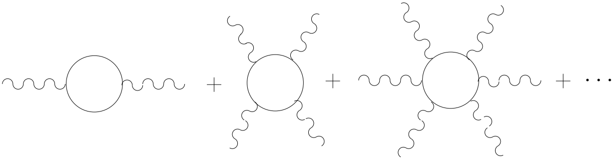
$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

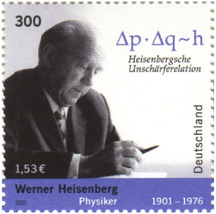
> *Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*

- *suggests* perturbative expansion cannot be convergent

Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216





- 1-loop QED effective action in uniform emag field
- e.g., constant B field:

$$S = -\frac{B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)\exp\left[-\frac{m^2 s}{B}\right]$$

$$S = -\frac{B^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2B}{m^2}\right)^{2n+2}$$

Euler-Heisenberg Effective Action

- e.g., constant B field: characteristic factorial divergence

$$c_n = \frac{(-1)^{n+1}}{8} \sum_{k=1}^{\infty} \frac{1}{(k\,\pi)^{2n+4}}\,\Gamma(2n+2)$$

- recall Borel summation:

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\,\Gamma(\gamma\, n + \delta)$$

$$\rightarrow \quad f(g) \sim \frac{1}{\gamma}\int_0^\infty \frac{ds}{s}\left(\frac{1}{1+s}\right)\left(\frac{s}{\beta g}\right)^{\delta/\gamma}\exp\left[-\left(\frac{s}{\beta g}\right)^{1/\gamma}\right]$$

- reconstruct correct Borel transform:

$$\sum_{k=1}^{\infty} \frac{s}{k^2\pi^2(s^2+k^2\pi^2)} = -\frac{1}{2s^2}\left(\coth s - \frac{1}{s} - \frac{s}{3}\right)$$

B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \operatorname{Im} S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

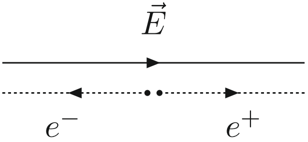
B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \text{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

Schwinger effect:



$$2eE\frac{\hbar}{mc} \sim 2mc^2$$

$$\Rightarrow$$

$$E_c \sim \frac{m^2 c^3}{e\hbar} \approx 10^{16}\text{V/cm}$$

WKB tunneling from Dirac sea

$\text{Im}\, S \to$ physical pair production rate

- Euler-Heisenberg series must be divergent

- scalar QED EH in self-dual background ($F = \pm\tilde{F}$):

$$S = \frac{F^2}{16\pi^2}\int_0^\infty \frac{dt}{t}\, e^{-t/F}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- Gaussian matrix model: $\lambda = g\,N$

$$\mathcal{F} = -\frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

- $c = 1$ String: $\lambda = g\,N$

$$\mathcal{F} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\, e^{-2\lambda\, t/g}\left(\frac{1}{\sin^2(t)} - \frac{1}{t^2} - \frac{1}{3}\right)$$

- Chern-Simons matrix model:

$$\mathcal{F} = -\frac{1}{4}\sum_{m\in\mathbb{Z}}\int_0^\infty \frac{dt}{t}\, e^{-2(\lambda+2\pi\, i\, m)\, t/g}\left(\frac{1}{\sinh^2(t)} - \frac{1}{t^2} + \frac{1}{3}\right)$$

de Sitter/ anti de Sitter effective actions (Das & GD, hep-th/0607168)

- explicit expressions (multiple gamma functions)

$$\mathcal{L}_{AdS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(AdS_d)} \left(\frac{K}{m^2}\right)^n$$

$$\mathcal{L}_{dS_d}(K) \sim \left(\frac{m^2}{4\pi}\right)^{d/2} \sum_n a_n^{(dS_d)} \left(\frac{K}{m^2}\right)^n$$

- changing sign of curvature: $a_n^{(AdS_d)} = (-1)^n a_n^{(dS_d)}$
- odd dimensions: convergent
- even dimensions: divergent

$$a_n^{(AdS_d)} \sim \frac{\mathcal{B}_{2n+d}}{n(2n+d)} \sim 2(-1)^n \frac{\Gamma(2n+d-1)}{(2\pi)^{2n+d}}$$

- pair production in dS_d with d even

QED/QCD effective action and the "Schwinger effect"

- formal definition:

$$\Gamma[A] = \ln \det \left(i \not{D} + m\right) \qquad\qquad D_\mu = \partial_\mu - i\frac{e}{\hbar c}A_\mu$$

- vacuum persistence amplitude

$$\langle O_{\rm out} \,|\, O_{\rm in}\rangle \equiv \exp\left(\frac{i}{\hbar}\,\Gamma[A]\right) = \exp\left(\frac{i}{\hbar}\left\{{\rm Re}(\Gamma) + i\,{\rm Im}(\Gamma)\right\}\right)$$

- encodes nonlinear properties of QED/QCD vacuum
- vacuum persistence probability

$$\left|\langle O_{\rm out} \,|\, O_{\rm in}\rangle\right|^2 = \exp\left(-\frac{2}{\hbar}\,{\rm Im}(\Gamma)\right) \approx 1 - \frac{2}{\hbar}\,{\rm Im}(\Gamma)$$

- probability of vacuum pair production $\approx \frac{2}{\hbar}\,{\rm Im}(\Gamma)$
- cf. Borel summation of perturbative series, & instantons

QED/QCD effective action

- encodes nonlinear properties of QED/QCD vacuum
- polarization tensor: $\frac{\delta^2\Gamma}{\delta A_\mu \delta A_\nu} \to \Pi_{\mu\nu}$
- Euler & Heisenberg (1935):

$$\epsilon_{ik} = \delta_{ik} + \frac{e^4\hbar}{45\pi m^4 c^7}\left[2\left(\vec{E}^2 - \vec{B}^2\right)\delta_{ik} + 7B_iB_k\right]$$

$$\mu_{ik} = \delta_{ik} + \frac{e^4\hbar}{45\pi m^4 c^7}\left[2\left(\vec{E}^2 - \vec{B}^2\right)\delta_{ik} - 7E_iE_k\right]$$

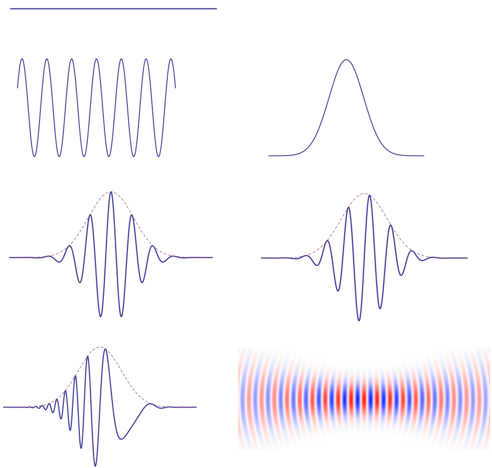
the electromagnetic properties of the vacuum can be described by a field-dependent electric and magnetic polarisability of empty space, which leads, for example, to refraction of light in electric fields or to a scattering of light by light *V. Weisskopf, 1936*

- PVLAS; ALPS, GammeV, BMV, OSQAR, ...

QFT in Extreme Background Fields: physical motivation

- perturbation theory is not applicable
- semiclassical/instanton/resurgence methods
- non-perturbative lattice methods
  - vacuum energy: mass generation; dark energy
  - beyond standard model: axion, ALP, dark matter searches
  - non-equilibrium QFT: e.g. quark-gluon-plasma
  - astrophysics: neutron stars, magnetars, black holes
  - cosmological particle production (Parker, Zeldovich)
  - Hawking radiation
  - back-reaction, cascading
  - ultimate electric field limit?

Schwinger Effect: Beyond Constant Background Fields



- constant field
- sinusoidal or single-pulse
- envelope pulse with sub-cycle structure; carrier-phase effect
- chirped pulse; Gaussian beam , ...

- envelopes and beyond require complex instantons
- physics: optimization and quantum control

Beyond Constant Background Fields

- Keldysh (1964): atomic ionization in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{\omega\sqrt{2mE_b}}{e\mathcal{E}}$
- WKB $\Rightarrow$ $P_{\text{ionization}} \sim \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\, g(\gamma)\right]$

$$P_{\text{ionization}} \sim \begin{cases} \exp\left[-\frac{4}{3}\frac{\sqrt{2m}E_b^{3/2}}{e\hbar\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\mathcal{E}}{2\omega\sqrt{2mE_b}}\right)^{2E_b/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- semi-classical instanton interpolates between non-perturbative 'tunneling ionization" and perturbative "multi-photon ionization"

Keldysh Approach in QED

Brézin/Itzykson, 1970; Popov, 1971

- Schwinger effect in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,c\,\omega}{e\,\mathcal{E}}$
- WKB $\Rightarrow$ $P_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\,g(\gamma)\right]$

$$P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \left(\frac{e\,\mathcal{E}}{\omega\,m\,c}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

- semi-classical instanton interpolates between non-perturbative ‘tunneling pair-production” and perturbative “multi-photon pair production”

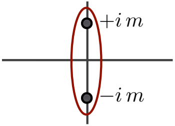
we will come back to this later ...

Scattering Picture of Particle Production

Feynman, Nambu, Fock, Brezin/Itzykson, Marinov/Popov, ...

- over-the-barrier scattering: e.g. scalar QED

$$-\ddot{\phi} - \left(p_3 - e\, A_3(t)\right)^2 \phi = (m^2 + p_\perp^2)\phi$$



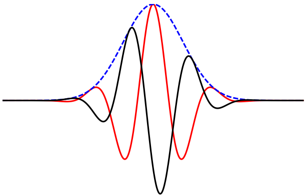
- pair production probability: $P \approx \int d^3p\, |b_p|^2$
- imaginary time method

$$|b_p|^2 \approx \exp\left[-2\,\mathrm{Im}\oint dt\,\sqrt{m^2 + p_\perp^2 + (p_3 - eA_3(t))^2}\right]$$

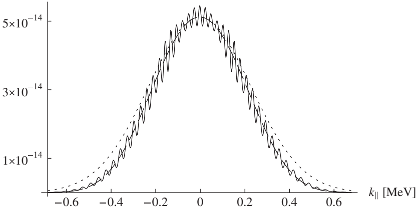
- more structured $E(t)$ involve quantum interference

Carrier Phase Effect

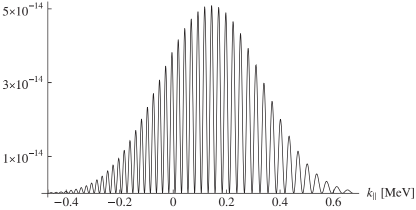
$$E(t) = \mathcal{E}\, \exp\left(-\frac{t^2}{\tau^2}\right)\, \cos\left(\omega t + \varphi\right)$$



- sensitivity to carrier phase φ ?



$\varphi = 0$



$\varphi = \frac{\pi}{2}$

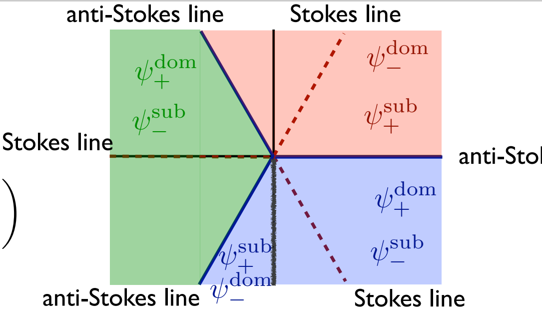
Particle Production as the Stokes Phenomenon

Dumlu, GD, 2010

$$\hbar^2 \psi'' + Q\,\psi = 0$$

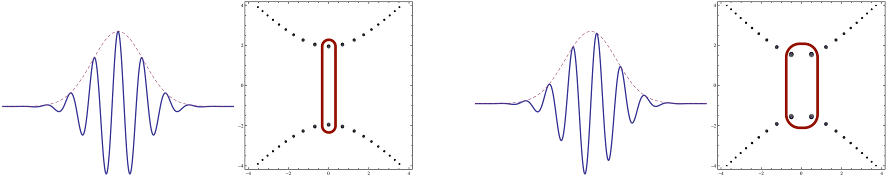
$$\downarrow$$

$$\psi_\pm = \frac{1}{Q^{1/4}} \exp\left(\pm \frac{i}{\hbar} \int^z Q^{1/2} \right)$$

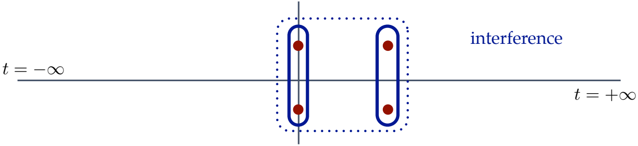


- suppose Q has simple zero at $z = 0$: $\quad \psi_\pm \sim \frac{\exp\left(\pm i\, z^{3/2}\right)}{z^{1/4}}$
- WKB solutions defined locally, inside Stokes wedges
- propagating from $t = -\infty$ to $t = +\infty$ necessarily crosses Stokes lines
- "birth" of a new exponential = particle production
- multiple sets of turning points $\Rightarrow$ quantum interference

Carrier Phase Effect from the Stokes Phenomenon



- interference produces momentum spectrum structure

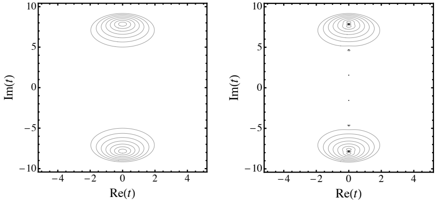


$$P \approx 4\sin^2(\theta)\, e^{-2\,\mathrm{Im}W} \qquad \theta\text{: interference phase}$$

- double-slit interference, in time domain, from vacuum
- Ramsey effect: N alternating sign pulses $\Rightarrow$ N-slit system $\Rightarrow$ coherent N^2 enhancement

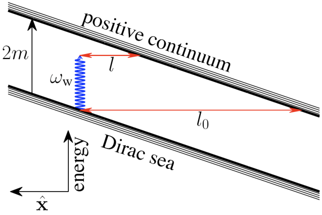
Akkermans, GD, 2012

- optical+X-ray laser pulse: $E(t) = \mathcal{E}_O(\Omega t) + \varepsilon_X(\omega t)$
- exponential enhancement due to new turning points



- "multi-photon assisted tunneling"

lowers Schwinger critical field from $10^{29} W/cm^2$ to $\sim 10^{25} W/cm^2$



(Di Piazza et al, 2009)

To maintain the relativistic invariance we describe a trajectory in space-time by giving the four variables $x_\mu(u)$ as functions of some fifth parameter (somewhat analogous to the proper-time) *Feynman, 1950*

- worldline representation of effective action

$$\Gamma = -\int d^4x \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_x \mathcal{D}x \, \exp\left[-\int_0^T d\tau \left(\dot{x}_\mu^2 + A_\mu \dot{x}_\mu\right)\right]$$

- double-steepest descents approximation:
- worldline instantons: $\ddot{x}_\mu = F_{\mu\nu}(x)\,\dot{x}_\nu$
- proper-time integral: $\frac{\partial S(T)}{\partial T} = -m^2$

$$\mathrm{Im}\,\Gamma \approx \sum_{\text{instantons}} e^{-S_{\text{instanton}}(m^2)}$$

- multiple turning point pairs $\Rightarrow$ complex instantons

Divergence of derivative expansion

GD, T. Hall, hep-th/9902064

- time-dependent E field: $E(t) = E\,\mathrm{sech}^2\,(t/\tau)$

$$\Gamma = -\frac{m^4}{8\pi^{3/2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(m\lambda)^{2j}} \sum_{k=2}^{\infty} (-1)^k \left(\frac{2E}{m^2}\right)^{2k} \frac{\Gamma(2k+j)\Gamma(2k+j-2)\mathcal{B}_{2k+2j}}{j!(2k)!\Gamma(2k+j+\frac{1}{2})}$$

- Borel sum perturbative expansion: large k (j fixed):

$$c_k^{(j)} \sim 2\frac{\Gamma(2k+3j-\frac{1}{2})}{(2\pi)^{2j+2k+2}}$$

$$\mathrm{Im}\,\Gamma^{(j)} \sim \exp\left[-\frac{m^2\pi}{E}\right] \frac{1}{j!} \left(\frac{m^4\pi}{4\tau^2 E^3}\right)^j$$

- resum derivative expansion

$$\mathrm{Im}\,\Gamma \sim \exp\left[-\frac{m^2\pi}{E}\left(1 - \frac{1}{4}\left(\frac{m}{E\tau}\right)^2 + \dots\right)\right]$$

Divergence of derivative expansion

- Borel sum derivative expansion: large j (k fixed):

$$c_j^{(k)} \sim 2^{\frac{9}{2}-2k} \frac{\Gamma(2j+4k-\frac{5}{2})}{(2\pi)^{2j+2k}}$$

$$\mathrm{Im}\,\Gamma^{(k)} \sim \frac{(2\pi E\tau^2)^{2k}}{(2k)!}\, e^{-2\pi m\tau}$$

- resum perturbative expansion:

$$\mathrm{Im}\,\Gamma \sim \exp\left[-2\pi m\tau\left(1-\frac{E\tau}{m}+\dots\right)\right]$$

- compare:

$$\mathrm{Im}\,\Gamma \sim \exp\left[-\frac{m^2\pi}{E}\left(1-\frac{1}{4}\left(\frac{m}{E\tau}\right)^2+\dots\right)\right]$$

- different limits of full: $\mathrm{Im}\,\Gamma \sim \exp\left[-\frac{m^2\pi}{E}\, g\left(\frac{m}{E\tau}\right)\right]$
- derivative expansion must be divergent

QFT: Renormalons

reviews: Beneke, 1998; Shifman, 2014

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

$$c_n \sim (\pm 1)^n \frac{n!}{(2S)^n}$$

QFT: new physical effects occur, due to running of couplings with momentum

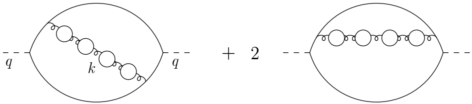
- faster source of divergence: "renormalons"

$$c_n \sim (\pm 1)^n \frac{\beta_0^n \, n!}{(2S)^n}$$

- both positive and negative Borel poles

Renormalons

- Adler function in QED: $D(Q^2) = -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}$



- bubble-chains, momentum $k \to$ interpolating expression

$$D(Q^2) = Q^2 \int_0^\infty \frac{k^2\, d(k^2)}{(k^2 + Q^2)^3} \frac{\alpha_s(Q^2)}{1 - \frac{\beta_0\, \alpha_s(Q^2)}{4\pi} \ln(Q^2/k^2)}$$

- running coupling $\alpha_s(k^2)$:

$$\alpha_s(k^2) = \frac{\alpha_s(Q^2)}{1 - \frac{\beta_0\, \alpha_s(Q^2)}{4\pi} \ln(Q^2/k^2)}$$

- β_0= first beta-function coefficient

- $\alpha_s(Q^2)$ expansion has factorial divergences from both small & large k^2

Renormalons

- split k^2 integral: $0 \leq k^2 \leq Q^2$ and $Q^2 \leq k^2 \leq \infty$
- low-momentum: $t = 2 \ln \frac{Q^2}{k^2}$; high momentum: $t = \ln \frac{k^2}{Q^2}$

$$\begin{aligned} D &= \frac{1}{Q^4} \int_0^{Q^2} dk^2 \frac{k^2 \alpha_s(k^2)}{(1+k^2/Q^2)^3} + \frac{1}{Q^4} \int_{Q^2}^{\infty} dk^2 \frac{k^2 \alpha_s(k^2)}{(1+k^2/Q^2)^3} \\ &= \frac{\alpha_s(Q^2)}{2} \int_0^{\infty} dt\, e^{-t} \sum_{a=0}^{\infty} \left\{ \frac{(-1)^a (1+a)}{\left(1 - \frac{\beta_0 \alpha_s}{4\pi} \frac{t}{(a+2)}\right)} + \frac{(-1)^a (2+a)}{\left(1 + \frac{\beta_0 \alpha_s}{4\pi} \frac{t}{(a+1)}\right)} \right\} \end{aligned}$$

- in Borel form, with poles on both $\mathbb{R}^{\pm}$

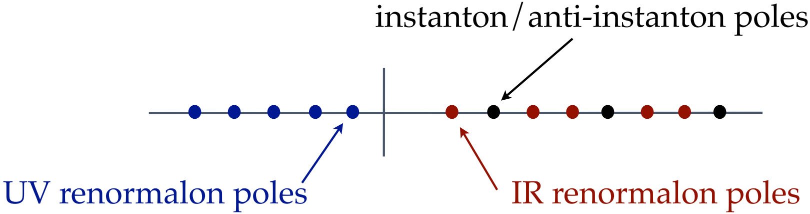
$$\begin{aligned} \text{IR}: \quad t_a^{\text{IR}} &= \left\{ \frac{8\pi}{\beta_0 \alpha_s}, \frac{12\pi}{\beta_0 \alpha_s}, \frac{16\pi}{\beta_0 \alpha_s}, \dots, \right\} \\ \text{UV}: \quad t_a^{\text{UV}} &= \left\{ -\frac{4\pi}{\beta_0 \alpha_s}, -\frac{8\pi}{\beta_0 \alpha_s}, -\frac{12\pi}{\beta_0 \alpha_s}, \dots, \right\} \end{aligned}$$

- key physics question: does the weakly-coupled theory "know enough" to extend into the strongly coupled region ?

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow$ $\pm i\, e^{-\frac{2S}{\beta_0 g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow$ $\pm i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

Lecture 3

- BZJ cancellation in 2d $\mathbb{CP}^{N-1}$ theory
- why resurgence?
- uniform WKB
- path integral interpretation: functional Darboux theorem
- thimbles and analytic continuation of path integrals

IR Renormalon Puzzle in Asymptotically Free QFT

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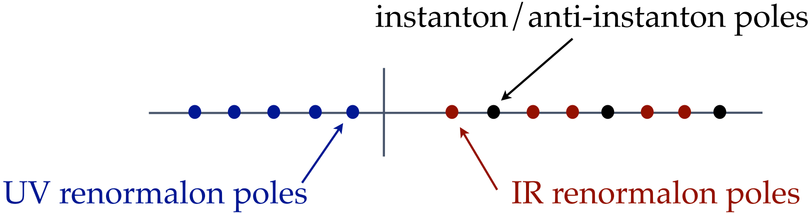

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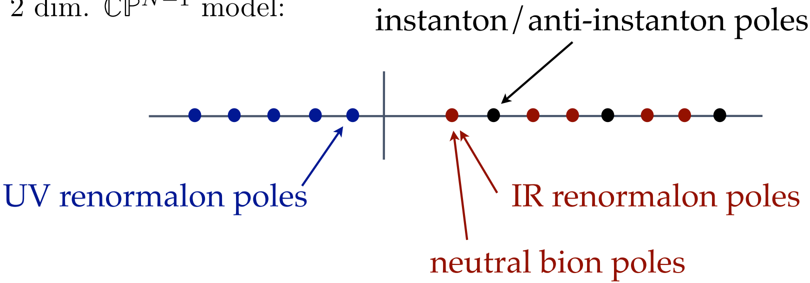
IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

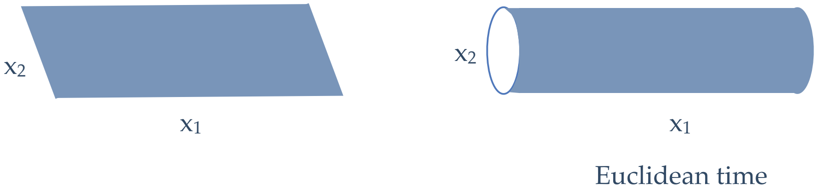
- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



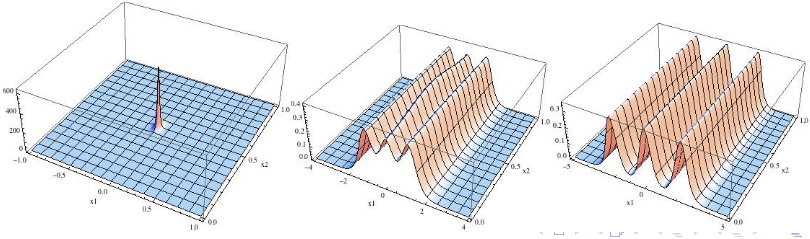
cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

Topological Molecules in Spatially Compactified Theories

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification: $\mathbb{R}^2 \to \mathbb{S}^1_L \times \mathbb{R}^1$

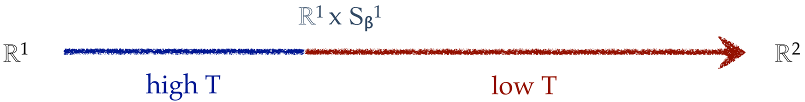


$\mathbb{Z}_N$ twist: instantons fractionalize: $S_{\rm inst} \longrightarrow \frac{S_{\rm inst}}{N} = \frac{S_{\rm inst}}{\beta_0}$



Topological Molecules in Spatially Compactified Theories

temporal conpactification: information only about deconfined phase



spatial compactification: semi-classical small L regime continuously connected to large L:

principle of continuity

Topological Molecules in Spatially Compactified Theories

- weak-coupling semi-classical analysis
- non-perturbative: kink-instantons: $\mathcal{I}_i$, $i = 1, 2, \ldots, N$
- bions: topological molecules of $\mathcal{I}\bar{\mathcal{I}}$
- "orientation" dependence of $\mathcal{I}\bar{\mathcal{I}}$ interaction:
- charged bions $\mathcal{B}_{ij} = [\mathcal{I}_i \bar{\mathcal{I}}_j]$: repulsive bosonic interaction
- neutral bions $\mathcal{B}_{ii} = [\mathcal{I}_i \bar{\mathcal{I}}_i]$: attractive bosonic interaction
- instanton/anti-instanton amplitude is ambiguous:

$$\left[\mathcal{I}_i \bar{\mathcal{I}}_i\right]_{\pm} = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right) \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

Perturbative Analysis

- weak-coupling semi-classical analysis
- perturbative $\rightarrow$ effective QM problem (Mathieu)
- perturbation theory diverges & non-Borel summable
- perturbative sector: lateral Borel summation

$$B_\pm \mathcal{E}(g^2) = \frac{1}{g^2} \int_{C_\pm} dt \, B\mathcal{E}(t) \, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi \, \frac{16}{g^2 \, N} \, e^{-\frac{8\pi}{g^2 \, N}}$$

- compare:

$$\left[\mathcal{I}_i \bar{\mathcal{I}}_i\right]_\pm = \left(\ln\left(\frac{g^2 \, N}{8\pi}\right) - \gamma \right) \frac{16}{g^2 \, N} \, e^{-\frac{8\pi}{g^2 \, N}} \pm i\pi \, \frac{16}{g^2 \, N} \, e^{-\frac{8\pi}{g^2 \, N}}$$

exact ("BZJ") cancellation !

explicit application of resurgence to nontrivial QFT

The Bigger Picture

Q: should we expect resurgent behavior in QM & QFT?

QM uniform WKB $\Rightarrow$

(i) trans-series structure is generic
(ii) all multi-instanton effects encoded in perturbation theory

(GD, Ünsal, 1306.4405, 1401.5202)

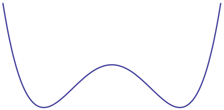
Q: what is behind this resurgent structure ?

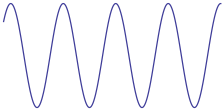
- basic property of all-orders steepest descents integrals

Q: could this extend to (path) functional integrals ?

Uniform WKB & Resurgent Trans-Series (1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi + \frac{V(g\,x)}{g^2}\psi = E\,\psi \rightarrow -g^4\,\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\,E\,\psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

$\Rightarrow$ uniform WKB with parabolic cylinder functions

- ansatz (with parameter ν): $\psi(y) = \dfrac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$

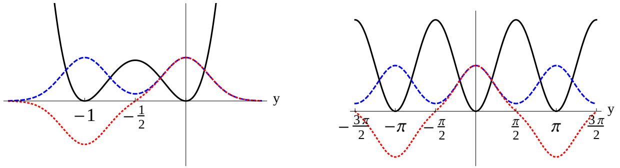
"similar looking equations have similar looking solutions"

Uniform WKB & Resurgent Trans-Series

- perturbative expansion for E and $u(y)$:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

- $\nu = N$: usual perturbation theory (not Borel summable)
- global analysis $\Rightarrow$ boundary conditions:



- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow$ $g^2 \to e^{\pm i\,\epsilon}\, g^2$
- trans-series encodes analytic properties of D_ν

  $\Rightarrow$ generic and universal

$$D_\nu(z) \sim z^\nu\, e^{-z^2/4}\,(1+\dots) + e^{\pm i\pi\nu}\,\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\, z^{-1-\nu}\, e^{z^2/4}\,(1+\dots)$$

$\longrightarrow$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}\,\mathcal{P}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}$):

$$\nu \;=\; N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{P}(N, g^2)}{N!}\,\xi$$

$$-\frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{P}\frac{\partial\mathcal{P}}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right) - \psi(N+1)\right)\mathcal{P}^2\right]\xi^2 + O(\xi^3)$$

- insert: $E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$ $\Rightarrow$ trans-series

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura: generate *entire trans-series* from

(i) perturbative expansion $E = E(N, g^2)$
(ii) single-instanton fluctuation function $\mathcal{P}(N, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

Connecting Perturbative and Non-Perturbative Sector

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(i) perturbative expansion $E = E(N, g^2)$
(ii) single-instanton fluctuation function $\mathcal{P}(N, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

in fact ... (GD, Ünsal, 1306.4405, 1401.5202)

$$\mathcal{P}(N, g^2) = \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4} \left(\frac{\partial E(N, g^2)}{\partial N} - 1 + \frac{\left(N + \frac{1}{2}\right) g^2}{S} \right)\right]$$

$\Rightarrow$ perturbation theory $E(N, g^2)$ encodes everything !

Connecting Perturbative and Non-Perturbative Sector

e.g. double-well potential: $B \equiv N + \frac{1}{2}$

$$E(N, g^2) = B - g^2\left(3B^2 + \frac{1}{4}\right) - g^4\left(17B^3 + \frac{19}{4}B\right) - g^6\left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - \dots$$

- non-perturbative function $(\mathcal{P} \sim (\dots)\exp[-A/2])$:

$$A(N, g^2) = \frac{1}{3g^2} + g^2\left(17B^2 + \frac{19}{12}\right) + g^4\left(125B^3 + \frac{153B}{4}\right) + g^6\left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) +$$

- simple relation:

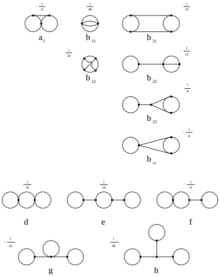
$$\frac{\partial E}{\partial B} = -3g^2\left(2B - g^2\frac{\partial A}{\partial g^2}\right)$$

Resurgence at work

• fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle determined by those about the vacuum saddle, to all fluctuation orders

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle determined by those about the vacuum saddle, to all fluctuation orders



- fluctuation about $\mathcal{I}$ for double-well:

2-loop (Shuryak/Wöhler, 1994); 3-loop

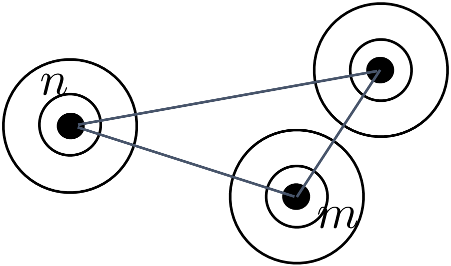
(Escobar-Ruiz/Shuryak/Turbiner, arXiv:1501.03993)

$$E/S/T: \quad e^{-\frac{S_0}{g}}\left[1-\frac{71}{72}\,g-0.607535\,g^2-\dots\right]$$

$$\text{D/Ü}: \quad e^{-\frac{S_0}{g}}\left[1+\frac{1}{72}\,g\left(-102N^2-174N-71\right)\right.$$

$$\left.+\frac{1}{10368}g^2\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\dots\right]$$

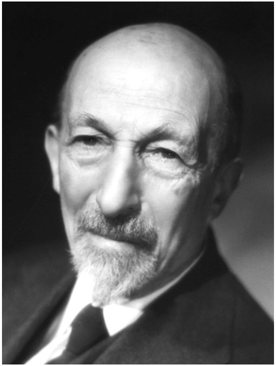
all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



why ? turn to path integrals ....

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard, 1865 - 1963



All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)} \left(\frac{F_{nm}}{v} \right)$$

- exact resurgent relation between fluctuations about $n^{\rm th}$ saddle and about neighboring saddles m

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \ldots \right.$$

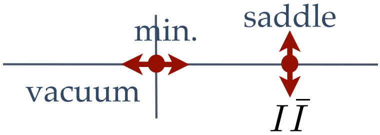
- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

All-Orders Steepest Descents: Darboux Theorem

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz\, e^{-\frac{1}{g^2}\sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$
$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right.$$

- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \dots\right)$$

- fluctuations about the two saddles are explicitly related could something like this work for path integrals?

"functional Darboux theorem" ?

- multi-dimensional case is already non-trivial and interesting

Pham (1965); Delabaere/Howls (2002)

- Picard-Lefschetz theory

- do a computation to see what happens ...

- periodic potential: $V(x) = \frac{1}{g^2} \sin^2(g\, x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots\right)$$

Resurgence in Path Integrals

(GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\, e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Complex Instantons

- recall complex instantons in non-perturbative imaginary part of QED effective action
- worldline instantons are Lefschetz thimbles
- how to compute them efficiently ?

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int \left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int \left|\frac{\delta S}{\delta A}\right|^2$$

- Chern-Simons theory (Witten 2001)
- comparison with complex Langevin (Aarts 2013, ...)
- lattice (Tokyo/RIKEN, Aurora, 2013): Bose-gas ✓

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Monte Carlo simulations on the Lefschetz thimble: Taming the sign problem

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CRISTOFORETTI *et al.*

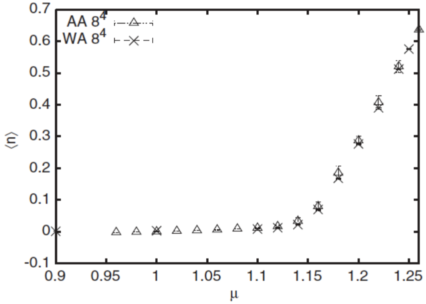
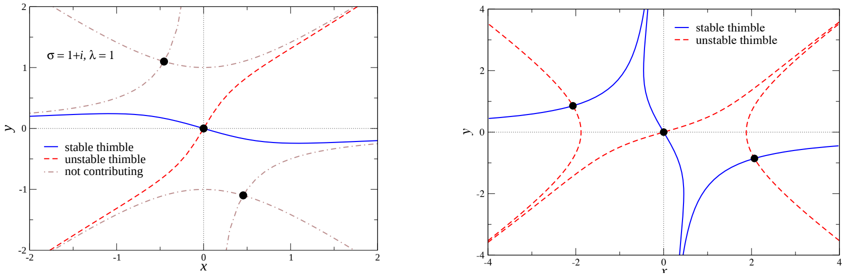


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA) presented here, for the lattice $V = 8^4$. We thank C. Gattringer and T. Kloiber for providing us their results.

Thimbles, Gradient Flow and Resurgence

$$Z = \int_{-\infty}^{\infty} dx \, \exp\left[-\left(\frac{\sigma}{2}x^2 + \frac{x^4}{4}\right)\right]$$

(Aarts, 2013; GD, Unsal, ...)



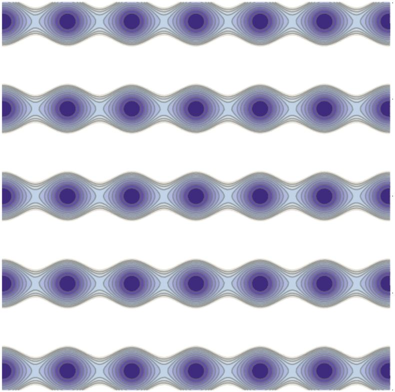
- contributing thimbles change with phase of σ
- need all three thimbles when $Re[\sigma] < 0$
- integrals along thimbles are related (resurgence)
- resurgence: preferred unique "field" choice

Ghost Instantons: Analytic Continuation of Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons



instanton actions:

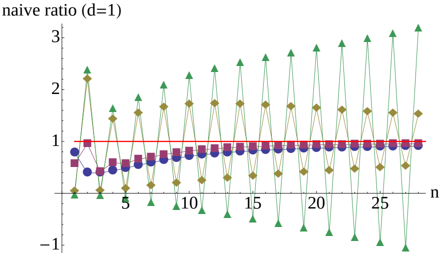
$$S_{\mathcal{I}}(m) = \frac{2\arcsin(\sqrt{m})}{\sqrt{m(1-m)}}$$

$$S_{\mathcal{G}}(m) = \frac{-2\arcsin(\sqrt{1-m})}{\sqrt{m(1-m)}}$$

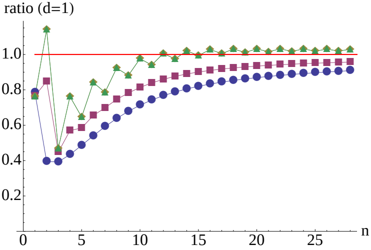
Ghost Instantons: Analytic Continuation of Path Integrals

- large order growth of perturbation theory:

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons



with ghost instantons

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

Non-perturbative Physics Without Instantons

Dabrowski, GD, 1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127, 1403.1277

- $O(N)$ & principal chiral model have no instantons !
- Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, principal chiral model, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Lecture 4

- connecting weak-coupling to strong-coupling
- resurgence and localization: some examples
- $\mathcal{N} = 2$ SUSY gauge theories and all-orders WKB
- quantum geometry

Connecting weak and strong coupling

main physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

. . . even if the degrees of freedom re-organize themselves in a very non-trivial way?

classical asymptotics is clearly not enough: is resurgent asymptotics enough?

Connecting weak and strong coupling

- often, weak coupling expansions are divergent, but strong-coupling expansions are convergent (generic behavior for special functions)

- e.g. Euler-Heisenberg

$$\Gamma(B) \sim -\frac{m^4}{8\pi^2} \sum_{n=0}^{\infty} \frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)} \left(\frac{2eB}{m^2}\right)^{2n+4}$$

$$\begin{aligned} \Gamma(B) \quad = \quad & \frac{(eB)^2}{2\pi^2} \left\{ -\frac{1}{12} + \zeta'(-1) - \frac{m^2}{4eB} + \frac{3}{4}\left(\frac{m^2}{2eB}\right)^2 - \frac{m^2}{4eB} \ln(2\pi) \right. \\ & + \left[-\frac{1}{12} + \frac{m^2}{4eB} - \frac{1}{2}\left(\frac{m^2}{2eB}\right)^2 \right] \ln\left(\frac{m^2}{2eB}\right) - \frac{\gamma}{2}\left(\frac{m^2}{2eB}\right)^2 \\ & \left. + \frac{m^2}{2eB}\left(1 - \ln\left(\frac{m^2}{2eB}\right)\right) + \sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n(n+1)} \left(\frac{m^2}{2eB}\right)^{n+1} \right\} \end{aligned}$$

(Drukker et al, 1007.3837; Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics

- 3d Chern-Simons on $\mathbb{S}^3$ $\to$ matrix model

$$Z_{CS}(N,g)=\frac{1}{\text{vol}(U(N))}\int dM \exp\left[-\frac{1}{g}\,\text{tr}\left(\frac{1}{2}\left(\ln M\right)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G=U(N)_k\times U(N)_{-k}$

$$Z_{ABJM}(N,k)=\sum_{\sigma\in S_N}\frac{(-1)^{\epsilon(\sigma)}}{N!}\int\prod_{i=1}^{N}\frac{dx_i}{2\pi k}\frac{1}{\prod_{i=1}^{N}2\text{ch}\left(\frac{x_i}{2}\right)\,\text{ch}\left(\frac{x_i-x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2)=\frac{1}{\text{vol}(U(N))}\int dM \exp\left[-\frac{1}{g^2}\,\text{tr}M^2\right]$$

Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- non-Borel-summable perturbation theory:

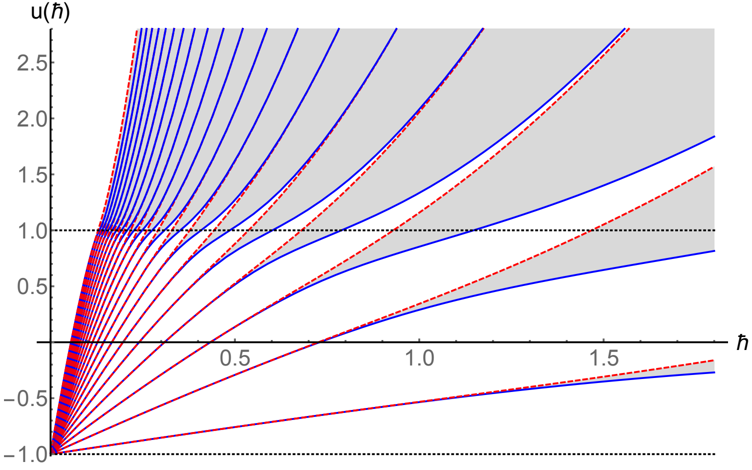
$$\begin{aligned} u(N,\hbar) \quad \sim \quad & -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right] \\ & -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3 + \frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots \end{aligned}$$

- energy is really a function of two variables:

$$u = u(N,\hbar)$$

Mathieu Equation Spectrum: ($\hbar$ plays role of g)

$$-\frac{\hbar^2}{2}\,\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$

- small N: divergent, non-Borel-summable:

$$u(N,\hbar) \sim -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\dots$$

- large N: convergent expansion:

$$u(N,\hbar)\sim\frac{\hbar^2}{8}\left(N^2+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 +\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\dots\right)$$

- different expansions and different degrees of freedom !

Small g and Large N

- often we study theories with both g and N
- 't Hooft limit: $\lambda \equiv N\, g$ fixed
- planar limit of QCD/YM: $J_n \sim n!$ but $J_n^{\rm planar} \sim c^n$
- e.g. Bessel functions:

$$Z_N\left(\frac{1}{g}\right) \equiv I_N\left(N\frac{1}{Ng}\right) \sim \begin{cases} \sqrt{\frac{g}{2\pi}}\, e^{1/g} & , \quad g \to 0,\ N \text{ fixed} \\ \frac{1}{\sqrt{2\pi N}}\left(\frac{e}{2Ng}\right)^N & , \quad N \to \infty,\ g \text{ fixed} \end{cases}$$

Small g and Large N

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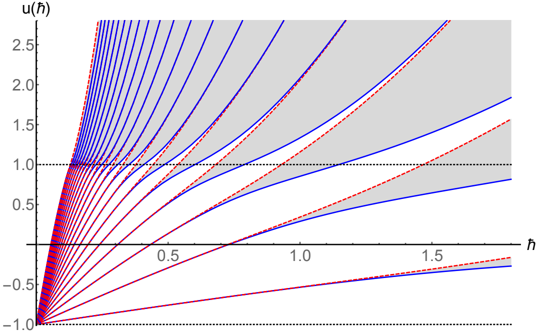
- uniform asymptotics:

$$Z_N\left(\frac{1}{g}\right) = I_N\left(N\frac{1}{Ng}\right) \sim \frac{\exp\left[\sqrt{N^2 + \frac{1}{g^2}}\right]}{\sqrt{2\pi}\left(N^2 + \frac{1}{g^2}\right)^{\frac{1}{4}}} \left(\frac{\frac{1}{N\,g}}{1 + \sqrt{1 + \frac{1}{(N\,g)^2}}}\right)^N$$

- analogue of Keldysh tunneling/multi-photon transition

Non-perturbative splittings



narrow bands: $\Delta u_N^{\text{band}} \sim \sqrt{\frac{2}{\pi}} \frac{2^{4(N+1)}}{N!} \left(\frac{2}{\hbar}\right)^{N-\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]$

narrow gaps: $\Delta u_N^{\text{gap}} \sim \frac{N\,\hbar^2}{2\pi} \left(\frac{e}{N\,\hbar}\right)^{2N}$

equal bands and gaps: $\Delta u_N^{\text{band}} \sim \Delta u_N^{\text{gap}} \sim O(\hbar)$

- recall Keldysh tunneling/multi-photon transition

Strong/weak coupling

what about a QFT in which the vacuum re-arranges itself in a non-trivial manner?

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- moduli parameter: $u = \langle \mathrm{tr}\,\Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{F}}{\partial a}$
- Nekrasov prepotential:

$$\mathcal{F}_{NS}(a,\hbar) = \mathcal{F}^{class.}(a,\hbar) + \mathcal{F}^{pert.}(a,\hbar) + \mathcal{F}^{inst.}(a,\hbar)$$

$$\mathcal{F}^{inst} \sim \frac{\hbar^2}{2\pi i}\left(\frac{\Lambda^4}{16a^4} + \frac{21\Lambda^8}{256a^8} + \dots\right) + \frac{\hbar^4}{2\pi i}\left(\frac{\Lambda^4}{64a^6} + \frac{219\Lambda^8}{2048a^{10}} + \dots\right) + \dots$$

$$\mathcal{F}^{class} + \mathcal{F}^{pert} \sim -\frac{a^2}{2\pi i}\log\frac{a^2}{\Lambda^2} - \frac{\hbar^2}{48\pi i}\log\frac{a^2}{2\Lambda^2} + \hbar^2\sum_{n=1}^{\infty} d_{2n}\left(\frac{\hbar}{a}\right)^{2n}$$

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- Mathieu equation:

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2\cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- all-orders WKB action: (Dunham, 1932)

$$a(u) = \frac{\sqrt{2}}{2\pi}\left(\int_{-\pi}^{\pi}\sqrt{u-V}\,dx - \frac{\hbar^2}{2^6}\int_{-\pi}^{\pi}\frac{(V')^2}{(u-V)^{5/2}}dx - \dots\right)$$

$$\Rightarrow \qquad a(u) = \sum_{n=0}^{\infty}\hbar^{2n}\,a_n(u)$$

- Bohr-Sommerfeld in large u (electric) region:

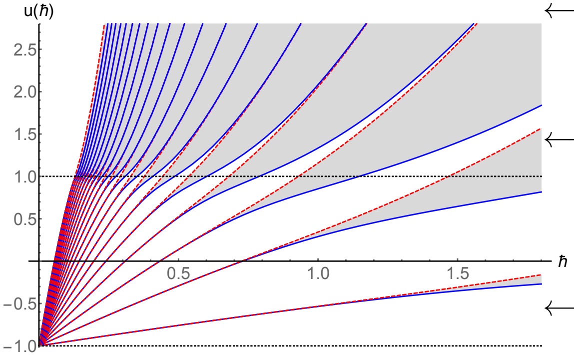
$$\text{invert} \quad a(u) = \frac{N}{2}\,\hbar \quad \Longrightarrow \quad u = u(N,\hbar) = u(a,\hbar)$$

- Matone relation:

$$u(a,\hbar) = \frac{i\pi}{2}\Lambda\frac{\partial\mathcal{F}_{NS}(a,\hbar)}{\partial\Lambda} - \frac{\hbar^2}{48}$$

Resurgence in $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ Theories (Başar, GD, 1501.05671)

- Mathieu & Lamé eqs encode Nekrasov prepotential



← electric sector (convergent)

← magnetic sector

← dyonic sector (divergent)

- resurgent WKB: $u = u(N, \hbar)$
- 't Hooft coupling: $\lambda \equiv N\,\hbar$
- very different physics for $\lambda \gg 1$, $\lambda \sim 1$, $\lambda \ll 1$

Divergent versus Convergent

- dyonic region (divergent, non-Borel-summable):

$$u(N,\hbar) \sim -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\dots$$

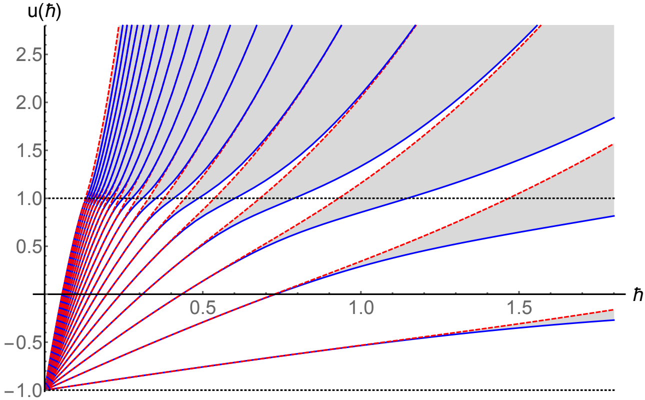
- electric region (convergent, but coefficients have poles):

$$u(N,\hbar)\sim\frac{\hbar^2}{8}\left(N^2+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 +\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\dots\right)$$

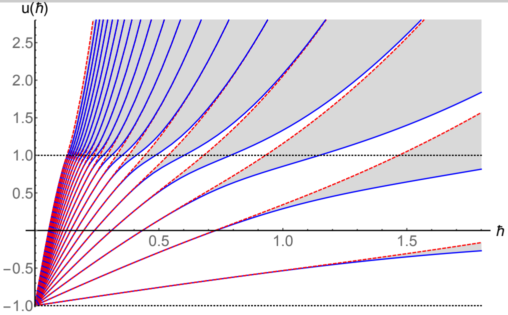
- different expansions and different degrees of freedom !

Uniform WKB and $\mathcal{N} = 2$ SUSY SU(2) (Başar, GD, 1501.05671)

- Bohr-Sommerfeld misses non-perturbative physics
- misses band and gap splittings
- smooth transition through magnetic region ?



Non-perturbative splittings



⟵ electric sector (convergent)

⟵ magnetic sector

⟵ dyonic sector (divergent)

dyonic: $$\Delta u_N^{\rm band} \sim \sqrt{\frac{2}{\pi}}\,\frac{2^{4(N+1)}}{N!}\left(\frac{2}{\hbar}\right)^{N-\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]$$

electric: $$\Delta u_N^{\rm gap} \sim \frac{N\,\hbar^2}{2\pi}\left(\frac{e}{N\,\hbar}\right)^{2N}$$

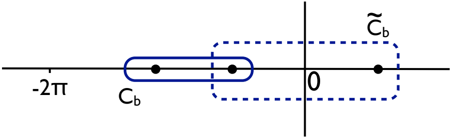
magnetic: $$\Delta u_N^{\rm band} \sim \Delta u_N^{\rm gap} \sim O(\hbar)$$

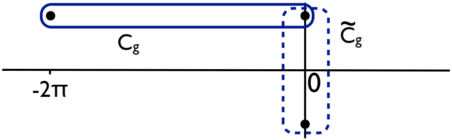
- recall Keldysh tunneling/multi-photon transition

Uniform WKB and $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- Bohr-Sommerfeld misses non-perturbative physics
- universal band/gap splitting: (Landau, Dykhne, Keller, ...)

$$\Delta u(N,\hbar) \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\,\exp\left[-\frac{2\pi}{\hbar}\,\mathrm{Im}\,a_D\right]$$





- dyonic sector: $\Delta u(N,\hbar) \sim \frac{64}{\sqrt{\pi}}\left(\frac{32}{\hbar}\right)^{N-\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right]$
- electric sector: $\Delta u(N,\hbar) \sim \frac{N\hbar^2}{2\pi}\left(\frac{e}{\hbar N}\right)^{2N}$
- magnetic sector: bands & gaps $\sim O(\hbar)$ (equal !)

Multi-instantons at strong coupling (!)

- dyonic region (divergent, non-Borel-summable):

$$
\begin{aligned}
u(N,\hbar) \quad \sim \quad & -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N + \frac{1}{2}\right)^2 + \frac{1}{4}\right] \\
& - \frac{\hbar^3}{16^2}\left[\left(N + \frac{1}{2}\right)^3 + \frac{3}{4}\left(N + \frac{1}{2}\right)\right] - \ldots
\end{aligned}
$$

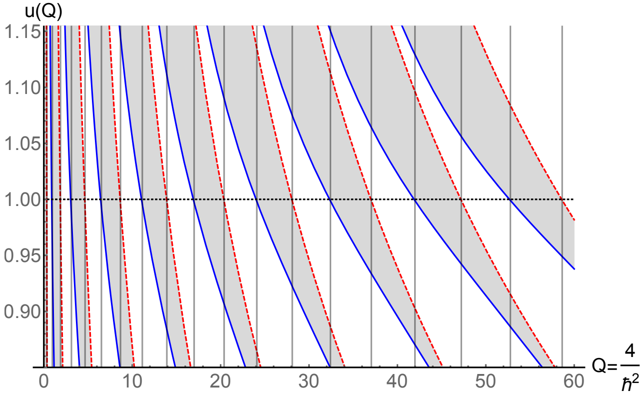
- electric region (convergent, but coefficients have poles):

$$
\begin{aligned}
u(N,\hbar) \sim \frac{\hbar^2}{8}\Bigg(& N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 \\
& + \frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12} + \ldots \Bigg)
\end{aligned}
$$

- multi-instanton structure in both sectors !

Magnetic region

- in this region instantons are large
- bands and gaps are of equal width



- degrees of freedom re-organize from tight-binding ‘atomic’ states to ‘nearly-free’ scattering states

Uniform WKB provides uniform analysis

- all-orders WKB action: (Dunham, 1932)

$$a = \sum_{n=0}^{\infty} \hbar^{2n} \, a_n(u)$$

- dyonic: expand $a_n(u)$ for $u \sim -1$; invert with $a = \frac{(N+\frac{1}{2})\hbar}{2}$
- electric: expand $a_n(u)$ for $u \gg 1$, & invert with $a = \frac{N\hbar}{2}$
- magnetic: expand $a_n(u)$ for $u \sim 1$, & invert with $a = \frac{(N+\frac{1}{2})\hbar}{2}$ or $a = \frac{N\hbar}{2}$
- Matone relation:

$$u(a, \hbar) = \frac{i\pi}{2} \Lambda \frac{\partial \mathcal{F}_{NS}(a, \hbar)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

Perturbative/Non-perturbative connection (Başar, GD, 1501.05671)

- Zinn-Justin: $B(u,\hbar)$, $A(u,\hbar)$ determine full trans-series
- GD, Ünsal: $u(B,\hbar)$ encodes $A(B,\hbar)$:

$$\frac{\partial u}{\partial B} = -\frac{\hbar}{16}\left(2B + \hbar\frac{\partial A}{\partial \hbar}\right)$$

- simple proof from Nekrasov $\mathcal{F}$ and Matone relation

$$u \sim \Lambda\frac{\partial \mathcal{F}}{\partial \Lambda} \quad \Rightarrow \quad \frac{\partial u}{\partial a} \sim \Lambda\frac{\partial}{\partial \Lambda}\frac{\partial \mathcal{F}}{\partial a} = \Lambda\frac{\partial a_D}{\partial \Lambda}$$

- identifications:

$$a \leftrightarrow \frac{\hbar}{2}B \quad , \quad a_D \leftrightarrow \frac{\hbar}{4\pi}A + \text{shift} \quad , \quad \Lambda \sim \frac{1}{\hbar}$$

- quantum geometry: $a(u,\hbar)$ and $a_D(u,\hbar)$ related
- **uniform** WKB spans electric/magnetic/dyonic sectors

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series 'encode' analytic continuation information
- expansions about different saddles are intimately related
- there is extra un-tapped 'magic' in perturbation theory

- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory

- fundamental property of steepest descents
- moral: go complex and consider all saddles, not just minima

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